Minimax Systems and Critical Point Theory

Martin Schechter

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To my wife, Deborah, our children,
our grandchildren (twenty four so far)
our great grandchildren (six so far)
and our extended family.
May they all enjoy many happy years.

Contents

Preface

Many problems in science involve the solving of differential equations or systems of differential equations. Moreover, many of these equations and systems come from variational considerations involving mappings (called functionals) into the real number system. As a simple example, consider the problem of finding a solution of

$$u''(x) = f(x, u(x)), \quad x \in I = [a, b], \tag{1}$$

under the conditions

$$u(a) = u(b) = 0. \tag{2}$$

Assume that the function $f(x, t)$ is continuous in $I \times \mathbb{R}$. The corresponding functional is

$$G(u) = \int_a^b [(u')^2 + 2F(x, u)]\, dx, \tag{3}$$

where

$$F(x, t) := \int_0^t f(x, s)\, ds.$$

It is easy to show that $u(x)$ is a solution of the problem (1), (2) if and only if it satisfies

$$G'(u) = 0. \tag{4}$$

Thus, in such cases, solutions of the equations or systems are critical points of the corresponding functional. As a result, anyone who is interested in obtaining solutions of the equations or systems is also interested in obtaining critical points of the corresponding functionals. The latter problem is the subject of this book.

The classical way of obtaining critical points was to search for maxima or minima. This is possible if the functional is bounded from above or below. However, when this is not the case, there is no organized way of finding critical points.

Linking theory is an attempt to "level the playing field," i.e., to find a substitute for semiboundedness. It finds a pair of subsets A, B of the underlying space that allow the functional to have the same advantages as semibounded functionals if the subsets separate the functional. They separate a functional G if

$$\sup_A G < \inf_B G. \tag{5}$$

This is the theme of the book [122], which records much of the work of researchers on this approach up to that time.

There are several methods of obtaining linking sets (some will be outlined later in Chapters 3 and 6). The purpose of the present volume is to unify some of these approaches and to study results and applications that were obtained since the publication of [122].

The underlying theme is to consider **minimax systems** depending on a set A. These are collections $\mathcal{K}$ of subsets such that if the functional G satisfies

$$\sup_A G < \inf_{K\in\mathcal{K}} \sup_K G, \tag{6}$$

then it has the same advantages as a semibounded functional. We show that the main approaches to linking can be combined by using minimax systems.

In Chapters 1 and 2, we define minimax systems and show what they can accomplish. We consider several variations and generalizations that are useful in applications.

In Chapter 3 we describe some methods used by researchers to obtain critical points of functionals, and we show that these results are contained in the theorems of Chapter 2. Various geometries are considered. In particular, the sandwich theorem of [143], [109], and [108] is generalized. This considers the following situation when N is a finite-dimensional subspace of a Hilbert space E and $M = N^\perp$. If G is a functional on E such that G is bounded from below on M and bounded from above on N, then G has the advantages of a semibounded functional. If both subspaces are infinite-dimensional, it may be necessary to impose additional assumptions on the functional G. This is discussed in Chapter 15.

In Chapter 4 we prove some theorems concerning differential equations in abstract spaces. These results are needed in proving the theorems of Chapter 2.

In Chapter 5 we give the proofs of the theorems of Chapter 2 using the results of Chapter 4.

In Chapter 6 we search for linking subsets. We believe in principle that we have found most, if not all, of them, at least in an abstract way. We produce two criteria for subsets to satisfy, one slightly stronger than the other. The weaker criterion is necessary for linking, while the stronger is sufficient.

In Chapter 7 we ask the question: Is there anything that can be done if one cannot find linking subsets that separate the functional? A surprising answer is yes. We show that there are subsets A, B that produce the same effect when they *do not separate the functional*, i.e., when they satisfy just the opposite:

$$-\infty < \inf_B G \leq \sup_A G < \infty. \tag{7}$$

We call such sets a sandwich pair. The reason is that the subspaces described in the sandwich theorem of Chapter 3 form a sandwich pair. We describe criteria for obtaining sandwich pairs.

In Chapter 8 we describe applications of the theories presented in Chapters 1–7. Some are more involved than those usually found in the literature. Other applications are given in Chapters 9–17.

In Chapter 9 we describe superlinear problems of the form

$$-\Delta u = f(x,u),\ x \in \Omega; \quad u = 0 \text{ on } \partial\Omega, \tag{8}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain whose boundary is a smooth manifold and $f(x,t)$ is a continuous function on $\bar{\Omega} \times \mathbb{R}$. The problem is called superlinear if $f(x,t)$ does not satisfy an inequality of the form

$$|f(x,t)| \leq C(|t|+1), \quad x \in \Omega,\ t \in \mathbb{R}.$$

Almost all researchers who studied the superlinear problem make the assumption that there are constants $\mu > 2,\ r \geq 0$ such that

$$0 < \mu F(x,t) \leq tf(x,t), \quad |t| \geq r. \tag{9}$$

This is a very convenient assumption, but it excludes a large number of superlinear problems. An assumption that is much weaker is: Either

$$F(x,t)/t^2 \to \infty \text{ as } t \to \infty$$

or

$$F(x,t)/t^2 \to \infty \text{ as } t \to -\infty.$$

We shall show that under this condition the boundary value problem

$$-\Delta u = \beta f(x,u),\ x \in \Omega; \quad u = 0 \text{ on } \partial\Omega, \tag{10}$$

has a nontrivial solution for almost every positive β. We require a stronger hypothesis to show that the statement is true for any particular β.

A fact of life which plagues researchers is that we can verify that subsets link only if at least one of them is contained in a finite dimensional manifold. In Chapter 10 we deal with this problem by adding an assumption to the functional. We try to make this assumption as weak as possible. Applications are given.

Chapters 11 and 14 deal with the Fučík spectrum. They deal with the situation concerning the semilinear problem

$$Au = f(x,u), \quad u \in D(A), \tag{11}$$

in which

$$\begin{aligned} f(x,t)/t \quad &\to \quad a \text{ a.e. as } t \to -\infty \\ &\to \quad b \text{ a.e. as } t \to +\infty, \end{aligned}$$

and

$$Au = bu^+ - au^-, \quad u^\pm = \max\{\pm u, 0\}, \tag{12}$$

has a nontrivial solution. We call the set Σ of those $(a,b) \in \mathbb{R}^2$ for which (12) has nontrivial solutions the Fučík spectrum of A. When $(a,b) \in \Sigma$, it is more difficult to

solve (11) than when $(a, b) \notin \Sigma$. Chapter 11 deals with solving (11) when $(a, b) \in \Sigma$. Moreover, not all $(a, b) \notin \Sigma$ are alike; some cause more difficulty than others. Chapter 14 deals with such points.

Chapters 12 and 13 are concerned with the multidimensional wave equation

$$\Box u \equiv u_{tt} - \Delta u = p(x, t, u), \tag{13}$$

$$u(x, t) = 0, \quad t \in \mathbb{R}, \ x \in \partial\Omega. \tag{14}$$

In Chapter 12 we study rotationally invariant solutions. In Chapter 13 we study the more general situation.

In Chapter 15 we discuss systems of equations for which the method of sandwich pairs would be ideal. However, sandwich pairs are plagued by the same problem as linking subsets, namely, that they cannot be verified as sandwich pairs unless at least one of the sets is contained in a finite-dimensional manifold. We show in Chapter 15 that they can both be infinite-dimensional if we add an assumption on G that is satisfied in most applications.

In Chapter 16 we show that in many situations, nonlinear boundary-value problems have multiple solutions.

Chapter 17 deals with second-order periodic systems of the form

$$-\ddot{x}(t) = \nabla_x V(t, x(t)), \tag{15}$$

where

$$x(t) = (x_1(t), \dots, x_n(t)). \tag{16}$$

We give several sets of conditions that imply the existence of solutions and conditions which imply the existence of nonconstant solutions.

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Chapter 1

Critical Points of Functionals

1.1 Introduction

Many problems arising in science and engineering call for the solving of the Euler–Lagrange equations of functionals, i.e., equations equivalent to

$$G'(u) = 0, \tag{1.1}$$

where $G(u)$ is a C^1-functional (usually representing the energy) arising from the given data. By this we mean that functions are solutions of the Euler–Lagrange equations of G iff they satisfy (1.1). (For various definitions of the derivative $G'(u)$ of G, cf., e.g., [127].) Solutions of (1.1) are called **critical points** of G. Thus, solving the Euler–Lagrange equations is tantamount to finding critical points of the corresponding functional.

As an illustration, the equation

$$-\Delta u(x) = f(x, u(x))$$

is the Euler–Lagrange equation of the functional

$$G(u) = \frac{1}{2}\|\nabla u\|^2 - \int F(x, u(x))\, dx$$

on an appropriate space, where

$$F(x, t) = \int_0^t f(x, s)\, ds, \tag{1.2}$$

and the norm is that of L^2. The variational approach to solving differential equations and systems has its roots in the calculus of variations. The original problem was to minimize or maximize a given functional. The approach was to obtain the Euler–Lagrange equations of the functional, solve them, and show that the solutions provided the required minimum or maximum. This worked well for one-dimensional problems. However, when it came to higher dimensions, it was recognized quite early that it was

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more difficult to solve the Euler–Lagrange equations than it was to find minima or maxima of functionals. Consequently, the approach was abandoned for many years. Eventually, when nonlinear partial differential equations and systems arose in applications and people were searching for solutions, they began to check if the equations and systems were the Euler–Lagrange equations of functionals. If so, a natural approach is to find critical points of the corresponding functionals. The problem is that there is no uniform way of finding them.

1.2 Extrema

The usual approach to finding critical points was to look for maxima or minima. Global extrema are the easiest to find, but they can exist only if the functional is semibounded. For instance, if the continuously differential functional G is bounded from below, then we can find a minimizing sequence $\{u_k\}$ such that

$$G(u_k) \to a = \inf G > -\infty. \tag{1.3}$$

If this series converges or has a convergent subsequence, we have a minimum. However, we have no guarantee that a subsequence will converge. Moreover, a minimizing sequence provides very little structure that can help one find a convergent subsequence. Indeed, one can find simple examples of functionals that are bounded below and have no minimum. For instance, the function

$$y(x) = e^x$$

has no minimum on $\mathbb{R}$ even though it is bounded from below.

1.3 Palais–Smale sequences

On the other hand, if the functional G is bounded from below, it can be shown that there is a sequence satisfying

$$G(u_k) \to a, \quad G'(u_k) \to 0. \tag{1.4}$$

(Actually, one can do better; see below.) If the sequence has a convergent subsequence, this will produce a minimum. Such a sequence is called a **Palais–Smale (PS) sequence.** The advantage of having a PS sequence instead of a minimizing sequence is that the additional information

$$G'(u_k) \to 0$$

helps one show that a PS sequence has a convergent subsequence more readily than one can show that a minimizing sequence has a convergent subsequence.

1.4 Cerami sequences

Actually, it can be shown that a C^1-functional G that is bounded from below has a **Cerami sequence,** a sequence satisfying

$$G(u_k) \to a, \quad (1 + \|u_k\|)G'(u_k) \to 0 \tag{1.5}$$

(cf. Theorem 3.21 and its corollary). A Cerami sequence says more. There are examples for which one can show that a Cerami sequence has a convergent subsequence, while one cannot do the same for the corresponding PS sequence.

1.5 Linking sets

When the functional is not semibounded, there is no clear way of obtaining critical points. In general, one would like to determine when a functional has PS or Cerami sequences. That is, one would like to find situations that give one the same advantages that one has in the case of semibounded functionals.

An idea that has been very successful is to find appropriate sets that **separate the functional.** By this we mean the following:

Definition 1.1. *Two sets A, B* **separate** *the functional $G(u)$ if*

$$a_0 := \sup_A G < b_0 := \inf_B G. \tag{1.6}$$

We would like to find sets A and B such that (1.6) will imply

$$\exists\, u \ : \ G(u) \geq b_0, \ G'(u) = 0. \tag{1.7}$$

This is too much to expect, since even semiboundedness alone does not imply the existence of a critical point. Consequently, we weaken our requirements and look for sets A, B such that (1.6) implies the existence of a PS sequence (1.4) with $a \geq b_0$. This leads to

Definition 1.2. *We shall say that the set A* **links** *the set B if (1.6) implies (1.4) with $a \geq b_0$ for every C^1-functional $G(u)$.*

Of course, (1.4) is a far cry from (1.7), but if, e.g., the sequence (1.4) has a convergent subsequence, then (1.4) implies (1.7). Whether or not (1.4) implies (1.7) is a property of the functional $G(u)$. We state this as

Definition 1.3. *We say that $G(u)$ satisfies the* **PS condition** *if (1.4) always implies (1.7).*

The usual way of verifying this is to show that every sequence satisfying (1.4) has a convergent subsequence (there are other ways).

All of this leads to

Theorem 1.4. *If G satisfies the PS condition and is separated by a pair of linking sets, then it has a critical point satisfying (1.7).*

This theorem cannot be applied until one identifies linking sets and functionals that satisfy the PS condition. Fortunately, they exist. We shall describe many of the known linking sets later in the book.

Among other things, we shall discuss when

$$a_0 := \sup_A G \le b_0 := \inf_B G \tag{1.8}$$

implies the existence of a PS sequence or a Cerami sequence.

If there do not exist linking sets that separate a functional, all is not lost. There exist sets that produce PS or Cerami sequences when they do not separate a functional. That is, we shall find sets such that

$$-\infty < b_0 := \inf_B G, \quad a_0 := \sup_A G < \infty \tag{1.9}$$

implies the existence of a PS sequence or a Cerami sequence.

1.6 Previous definitions of linking

In the earlier versions of linking, A is a compact set and is the "boundary" of a manifold S. They say that A links a set B if

$$A \cap B = \phi$$

and every continuous map φ from S to E that equals the identity on A must satisfy

$$\varphi(S) \cap B \neq \phi.$$

The underlying theorem is

Theorem 1.5. *If the set A links the set B and (1.6) holds, then there is a Palais–Smale sequence satisfying (1.4) with* $a \ge b_0$.

There are distinct disadvantages of this definition. First, it requires A to be compact. Second, it requires A to be the "boundary" of a manifold S. Third, linking depends on the manifold S. Finally, there is no possibility of symmetry in infinite-dimensional spaces, i.e., if A links B, then B cannot link A according to this definition.

Several methods have been used to circumvent these shortcomings and extend the definition of linking to cover cases when the old definition does not apply. Some of them are described in Chapter 3. The purpose of this volume is to present a uniform approach that includes most, if not all, theories of linking. It employs minimax systems and is presented in Chapter 2.

The outline of this book is as follows. In Chapter 5 we give the proofs of the theorems of Chapter 2. We use methods of solving differential equations in Banach

spaces described in Chapter 4. In Chapter 6 we show that our uniform method is almost identical to the general Definition 1.2 given above. In Chapter 7 we consider the case when there are no linking sets that separate the functional. We find pairs of sets (called sandwich pairs) that produce PS sequences (1.4) when (1.6) fails. In Chapters 8, 9, 11, 14, 16, and 17 we give some applications to partial differential equations of the methods presented in Chapters 1–7. In Chapters 10, 12, 13, and 15 we are concerned with the situation when A and B are both infinite-dimensional.

1.7 Notes and remarks

For an excellent review of critical point theory and linking, we refer to [96]. Previous definitions of linking theory are described in [8]. Other definitions of linking will be discussed in Chapters 2 and 3.

Chapter 2

Minimax Systems

2.1 Introduction

We have described how the variational approach to solving nonlinear problems eventually leads to the search for critical points of related functionals. In the case of semibounded functionals, one can look for extrema. Otherwise, one is forced to use other methods. As we mentioned, linking provides a useful tool. There are several approaches to linking. In this book we unify these approaches, providing one theory that works for all of them.

The idea is as follows. For each $A \subset E$, one wishes to find a collection $\mathcal{K}$ of sets K with the following properties.

1. If $G \in C^1(E, \mathbb{R})$ satisfies

$$a_0 := \sup_A G < a := \inf_{K \in \mathcal{K}} \sup_K G, \tag{2.1}$$

then there is a sequence $\{u_k\} \subset E$ such that (1.4) holds.

2. If

$$a_0 \leq \sup_{K \setminus A} G, \quad K \in \mathcal{K}, \tag{2.2}$$

then there is a sequence $\{u_k\} \subset E$ such that (1.4) holds.

3. If $A, B \subset E$ satisfy

$$A \cap B = \phi, \quad B \cap K \neq \phi, \quad K \in \mathcal{K}, \tag{2.3}$$

and

$$a_0 := \sup_A G < b_0 := \inf_B G, \tag{2.4}$$

then there is a sequence $\{u_k\} \subset E$ such that (1.4) holds.

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4. If

$$g_K = \{v \in K \backslash A : G(v) \geq a_0\} \neq \phi, \quad K \in \mathcal{K}, \tag{2.5}$$

then there is a sequence $\{u_k\} \subset E$ such that (1.4) holds.

In the next section we determine collections $\mathcal{K}$ that have these properties. Moreover, we shall show that all of these collections produce not only PS sequences (1.4), but also Cerami sequences (1.5) as well.

Proofs of the theorems of this chapter will be given in Chapter 5.

2.2 Definitions and theorems

We begin by studying C^1-functionals on a Banach space E.

Definition 2.1. *We shall say that a map $\varphi : E \to E$ is of class Λ if it is a homeomorphism onto E, and both φ, φ^{-1} are bounded on bounded sets.*

Definition 2.2. *For $A \subset E$, we define*

$$\Lambda(A) = \{\varphi \in \Lambda : \varphi(u) = u, \ u \in A\}.$$

Definition 2.3. *For a nonempty set $A \subset E$, we define a nonempty collection $\mathcal{K} = \mathcal{K}(A)$ of subsets $K \subset E$ to be a* **minimax system** *for A if it has the following property:*

$$\varphi(K) \in \mathcal{K}, \quad \varphi \in \Lambda(A), \ K \in \mathcal{K}.$$

Every nonempty set has a minimax system. We have

Theorem 2.4. *Let $\mathcal{K}$ be a minimax system for a nonempty subset A of E, and let G(u) be a C^1-functional on E. Define*

$$a := \inf_{K \in \mathcal{K}} \sup_K \ G, \tag{2.6}$$

and assume that a is finite and satisfies

$$a > a_0 := \sup_A G. \tag{2.7}$$

Let $\psi(t)$ be a positive, nonincreasing, locally Lipschitz continuous function on $[0, \infty)$ such that

$$\int_0^\infty \psi(r)\, dr = \infty. \tag{2.8}$$

Then there is a sequence $\{u_k\} \subset E$ such that

$$G(u_k) \to a, \quad G'(u_k)/\psi(\|u_k\|) \to 0. \tag{2.9}$$

Definition 2.5. *We shall call $\sigma(t) \in C([0, T] \times E, E)$, $T > 0$, a* **flow** *if $\sigma(0) = I$ and for each $t \in [0, T]$, $\sigma(t)$ is a homeomorphism of E onto itself.*

Theorem 2.6. *Let $\mathcal{K}$ be a minimax system for a nonempty set A and let G(u) be a C^1 functional on E such that*

$$g_K = \{v \in K \backslash A : G(v) \geq a_0\} \neq \phi, \quad K \in \mathcal{K}, \tag{2.10}$$

and

$$a = \inf_{\mathcal{K}} \sup_K G < \infty. \tag{2.11}$$

Assume that for any $b > a$, $K \in \mathcal{K}$ and flow $\sigma(t)$ satisfying

$$G(\sigma(t)v) < b, \quad v \in K, \ 0 < t \leq T, \tag{2.12}$$

and

$$G(\sigma(t)u) < a, \quad u \in A, \ 0 < t \leq T, \tag{2.13}$$

there is a $\tilde{K} \in \mathcal{K}$ such that

$$\tilde{K} \subset \bigcup_{t \in [0,T]} \sigma(t)A \ \cup \ \sigma(T)[E_b \cup K], \tag{2.14}$$

where

$$E_\alpha = \{u \in E : G(u) \leq \alpha\}. \tag{2.15}$$

Then the conclusions of Theorem 2.4 hold.

2.3 Linking subsets

We now show how linking can play a major role in finding critical points.

Definition 2.7. *We shall say that a set A in E* **links a set** $B \subset E$ **relative to a minimax system** $\mathcal{K}$ **for** A *if*

$$A \cap B = \phi \tag{2.16}$$

and

$$B \cap K \neq \phi, \quad K \in \mathcal{K}. \tag{2.17}$$

We shall say that A **links** *B* **[mm]** *if there is a minimax system $\mathcal{K}$ for A such that A links B relative to $\mathcal{K}$.*

Theorem 2.8. *Let $\mathcal{K}$ be a minimax system for a nonempty set A, and assume that there is a subset B of E such that A links B relative to $\mathcal{K}$. Assume that $G \in C^1(E, \mathbb{R})$ satisfies*

$$a_0 := \sup_A G < b_0 := \inf_B G \tag{2.18}$$

and that the quantity a given by (2.6) is finite. Then, for each positive, nonincreasing, locally Lipschitz continuous function $\psi(t)$ on $[0, \infty)$ satisfying (2.8), there is a sequence $\{u_k\} \subset E$ such that (2.9) holds.

Definition 2.9. *A subset A of a Banach space E* **links** *a subset B of E if, for every $G \in C^1(E, \mathbb{R})$ bounded on bounded sets and satisfying (2.18) there are a sequence $\{u_k\} \subset E$ and a constant a such that*

$$b_0 \le a < \infty \tag{2.19}$$

and (1.4) holds.

Definition 2.10. *A subset A of a Banach space E* **links a subset** *B* **of** *E* **strongly** *if, for every $G \in C^1(E, \mathbb{R})$ bounded on bounded sets and satisfying (2.18) and each positive, nonincreasing, locally Lipschitz continuous function $\psi(t)$ on $[0, \infty)$ satisfying (2.8), there is a sequence $\{u_k\} \subset E$ such that (2.9) and (2.19) hold.*

Theorem 2.11. *If A links B [mm], then it links B strongly.*

We note that Theorem 1.5 is a simple consequence of Theorem 2.4. In fact, we can let $\mathcal{K}$ be the collection

$$\mathcal{K} = \{\varphi(S) : \varphi \in C(S, E),\ \varphi(u) = u,\ u \in A\}.$$

It is easily checked that $\mathcal{K}$ is a minimax system for A. Moreover, if A links B in the old sense, then it links B [mm]. We can now apply Theorem 2.4.

In the following theorems, we allow $a = a_0$.

Theorem 2.12. *Let $\mathcal{K}$ be a minimax system for a nonempty set A, and assume that A links a subset B of E relative to $\mathcal{K}$. Let G be a C^1-functional satisfying (2.10), (2.11), and*

$$a_0 := \sup_A G \le b_0 := \inf_B G. \tag{2.20}$$

Assume that for any $b > a$, $K \in \mathcal{K}$, and flow $\sigma(t)$ satisfying (2.12), (2.13) and such that

$$\sigma(t)A \cap B = \phi, \quad t \in [0, T], \tag{2.21}$$

there is a $\tilde{K} \in \mathcal{K}$ such that (2.14) holds. Then the conclusions of Theorem 2.4 hold. Moreover, if $a_0 = a$, then there is a sequence satisfying (1.4) and

$$d(u_k, B) \to 0, \quad k \to \infty. \tag{2.22}$$

Theorem 2.13. *Let $\mathcal{K}$ be a minimax system for a nonempty set A satisfying*

$$K \backslash A \ne \phi, \quad K \in \mathcal{K}. \tag{2.23}$$

Let $G \in C^1$ satisfy (2.10) and (2.11). Assume that for any $K \in \mathcal{K}$ and flow $\sigma(t)$ satisfying (2.12), (2.13), one has $S(K) \in \mathcal{K}$, where

$$S(u) = \sigma(d(u, A))u, \quad u \in E. \tag{2.24}$$

Assume also that for each $K \in \mathcal{K}$, there is a $\rho > 0$ such that G(u) is Lipschitz continuous on

$$K_\rho = \{u \in K : d(u, A) \le \rho\}. \tag{2.25}$$

Then the conclusions of Theorem 2.4 hold.

Theorem 2.14. *Let $\mathcal{K}$ be a minimax system for a nonempty set A, and assume that there is a subset B of E such that A links B relative to $\mathcal{K}$. Assume that for any $K \in \mathcal{K}$ and flow $\sigma(t)$ satisfying (2.12), (2.13), one has $S(K) \in \mathcal{K}$, where S(u) is given by (2.24). Let G be a C^1-functional satisfying (2.11) and (2.20). Assume also that for each $K \in \mathcal{K}$ there is a $\rho > 0$ such that G(u) is Lipschitz continuous on K_ρ given by (2.25). Then the conclusions of Theorem 2.4 hold.*

Corollary 2.15. *Let $\mathcal{K}$ be a minimax system for a nonempty set A, and let $G \in C^1$ satisfy (2.10) and (2.11). Assume that for any $b > a$, $K \in \mathcal{K}$, and flow $\sigma(t)$ satisfying (2.12), (2.13), and (2.21) for*

$$B = \{u \in E \backslash A : G(u) \geq a_0\}, \tag{2.26}$$

there is a $\tilde{K} \in \mathcal{K}$ satisfying $\tilde{K} \subset \sigma(T)E_b$. Then the conclusions of Theorem 2.4 hold.

2.4 A variation

Theorem 2.16. *Let $\mathcal{K}$ be a minimax system for a nonempty subset A of E, and let G(u) be a C^1-functional on E. Define*

$$b := \sup_{K \in \mathcal{K}} \inf_K G, \tag{2.27}$$

and assume that b is finite and satisfies

$$b < b_0 := \inf_A G. \tag{2.28}$$

Then the conclusions of Theorem 2.4 hold with a replaced by b.

We also have the following. Let $A = \{0\}$. Then a minimax system for A can be constructed as follows. $\mathcal{K}$ consists of the boundaries of all bounded, open sets ω containing 0. That $\mathcal{K}$ is a minimax system for A follows from the fact that $0 \in \varphi(\omega)$ and $\partial\varphi(\omega) = \varphi(\partial\omega)$ for all such sets and $\varphi \in \Lambda(A)$. We have

Theorem 2.17. *Let b be defined by (2.27). Assume that $G(0) < b < \infty$. Then there is a sequence satisfying*

$$G(u_k) \to b, \quad G'(u_k)/\psi(\|u_k\|) \to 0. \tag{2.29}$$

Theorem 2.18. *Assume that $b < \infty$ and that there is an open set ω_0 containing 0 such that*

$$G(0) = \inf_{\partial\omega_0} G.$$

Then there is a sequence satisfying (2.29).

We also have the counterpart of these theorems for the quantity a given by (2.6).

Theorem 2.19. *Assume that $-\infty < a < G(0)$. Then the conclusions of Theorem 2.4 hold.*

Theorem 2.20. *Assume that $a > -\infty$ and that there is an open set ω_0 containing 0 such that*

$$G(0) = \sup_{\partial\omega_0} G.$$

Then the conclusions of Theorem 2.4 hold.

2.5 Weaker conditions

We now turn to the question as to what happens if some of the hypotheses of Theorem 2.12 do not hold. We are particularly interested in what happens when (2.20) is violated. In this case we let

$$B' := \{v \in B : G(v) < a_0\}. \tag{2.30}$$

Note that

$$B' = \phi \text{ iff } a_0 \le b_0.$$

We assume that $B' \neq \phi$. Let $\psi(t)$ be a positive, nonincreasing function on $[0, \infty)$ satisfying the hypotheses of Theorem 2.4 and such that

$$a_0 - b_0 < \int_\alpha^{R+\alpha} \psi(t)\,dt \tag{2.31}$$

for some finite $R \le d' := d(B', A)$, where $\alpha = d(0, B')$. We assume $d' > 0$. We have

Theorem 2.21. *Let G be a C^1-functional on E and $A, B \subset E$ be such that A links B [mm] and*

$$-\infty < b_0, \quad a < \infty. \tag{2.32}$$

Assume that for any $b > a$, $K \in \mathcal{K}$ and flow $\sigma(t)$ satisfying (2.12), (2.13), (2.21), there is a $\tilde{K} \in \mathcal{K}$ satisfying (2.14). Under the hypotheses given above, for each $\delta > 0$, there is a $u \in E$ such that

$$b_0 - \delta \le G(u) \le a + \delta, \quad \|G'(u)\| < \psi(d(u, B')). \tag{2.33}$$

We can also consider a slightly different version of Theorem 2.21. We consider the set

$$A'' := \{u \in A : G(u) > b_0\}, \tag{2.34}$$

and we note that $A'' = \phi$ iff $a_0 \le b_0$. We assume that ψ satisfies the hypotheses of Theorem 2.4 and

$$a_0 - b_0 < \int_\beta^{R+\beta} \psi(t)\,dt \tag{2.35}$$

holds for some finite $R \le d'' := d(A'', B)$, where $\beta = d(0, A'')$. Assume that $A'' \neq \phi$. We have

Theorem 2.22. *Assume that for any* $b < b_0$, $K \in \mathcal{K}$, *and flow* $\sigma(t)$ *satisfying*

$$G(\sigma(t)v) > b, \quad v \in K, \ t > 0, \tag{2.36}$$

$$G(\sigma(t)v) > b_0, \quad v \in B, \ t > 0, \tag{2.37}$$

and

$$\sigma(t)B \cap A = \phi, \quad t \in [0, T], \tag{2.38}$$

there is a $\tilde{K} \in \mathcal{K}$ *such that*

$$\tilde{K} \subset \bigcup_{t\in[0,T]} \sigma(t)B \ \cup \ \sigma(T)[E^b \cup K], \tag{2.39}$$

where

$$E^\alpha = \{u \in E : G(u) \geq \alpha\}. \tag{2.40}$$

If B links A [mm] and

$$-\infty < b_0, \quad a < \infty \tag{2.41}$$

hold, then, for each $\delta > 0$, *there is a* $u \in E$ *such that*

$$b_0 - \delta \leq G(u) \leq a + \delta, \quad \|G'(u)\| < \psi(d(u, A'')). \tag{2.42}$$

2.6 Some consequences

We now discuss some methods that follow from Theorems 2.21 and 2.22. Let $\{A_k, B_k\}$ be a sequence of pairs of subsets of E. Let $\mathcal{K}_k$ be a minimax system for A_k. For $G \in C^1(E, \mathbb{R})$, let

$$a_{k0} = \sup_{A_k} G, \quad b_{k0} = \inf_{B_k} G, \tag{2.43}$$

and

$$a_k = \inf_{K\in\mathcal{K}_k} \sup_K G. \tag{2.44}$$

We assume $a_k < \infty$ for each k. We define

$$B'_k := \{v \in B_k : G(v) < a_{k0}\}, \tag{2.45}$$

$$A''_k := \{u \in A_k : G(u) > b_{k0}\} \tag{2.46}$$

$$d'_k := d(A_k, B'_k), \quad d''_k := d(A''_k, B_k). \tag{2.47}$$

We have

Theorem 2.23. *Assume that A_k links B_k [mm] for each k, that*

$$d_k' \to \infty \text{ as } k \to \infty, \tag{2.48}$$

and for each k, there is a positive, nonincreasing function $\psi_k(t)$ on $[0, \infty)$ satisfying the hypotheses of Theorem 2.4 and such that

$$a_{k0} - b_{k0} < \int_{\alpha_k}^{R_k+\alpha_k} \psi_k(t)dt, \tag{2.49}$$

where $\alpha_k = d(0, B_k')$ and $R_k \le d_k'$. Then, under the hypotheses of Theorem 2.21, there is a sequence $\{u_k\} \subset E$ such that

$$b_{k0} - (1/k) \le G(u_k) \le a_k + (1/k) \tag{2.50}$$

and

$$\|G'(u_k)\| \le \psi_k(d(u_k, B_k')). \tag{2.51}$$

Theorem 2.24. *Assume that B_k links A_k [mm] for each k, that*

$$d_k'' \to \infty \text{ as } k \to \infty, \tag{2.52}$$

and that, for each k, there is a positive, nonincreasing function $\psi_k(t)$ on $[0, \infty)$ satisfying the hypotheses of Theorem 2.4 and such that

$$a_{k0} - b_{k0} < \int_{\beta_k}^{R_k+\beta_k} \psi_k(t)dt, \tag{2.53}$$

where $\beta_k = d(0, A_k'')$ and $R_k \le d_k''$. Then, under the hypotheses of Theorem 2.22, there is a sequence $\{u_k\} \subset E$ such that

$$b_{k0} - (1/k) \le G(u_k) \le a_k + (1/k) \tag{2.54}$$

and

$$\|G'(u_k)\| \le \psi_k(d(u_k, A_k'')). \tag{2.55}$$

We combine the proofs of Theorems 2.23 and 2.24.

Proof. For each k, take R_k equal to d_k' or d_k'', as the case may be. We may assume that $b_{k0} < a_{k0}$ for each k. Otherwise the conclusions of the theorems follow from Theorem 2.4. We can now apply Theorems 2.21 and 2.22 for each k to conclude that there is a $u_k \in E$ such that

$$b_{k0} - (1/k) \le G(u_k) \le a_k + (1/k),$$

and either

$$\|G'(u_k)\| < \psi_k(d(u_k, B_k'))$$

or

$$\|G'(u_k)\| < \psi_k(d(u_k, A_k'')),$$

as the case may be. $\square$

2.7 Notes and remarks

The concept of a minimax system is related to that of [74] (cf. Section 3.4). The first situation that allowed $a_0 = a$ appeared in [75]. The first to consider the case $\psi(t) = 1/(1 + |t|)$ was [39]. The first to consider the general case of arbitrary nonincreasing ψ was [107]. The theorems of Section 2.4 are due to [107]. The theorems of Sections 2.2, 2.3, 2.5, and 2.6 are due to [134].

Chapter 3

Examples of Minimax Systems

3.1 Introduction

In this chapter we present some linking methods which are special cases of linking with respect to minimax systems. We consider three useful methods and show that they provide minimax systems. We then consider examples and applications.

3.2 A method using homeomorphisms

One linking method can be described as follows. Let E be a Banach space and let Φ be the set of all continuous maps $\Gamma = \Gamma(t)$ from $E \times [0, 1]$ to E such that

1. $\Gamma(0) = I$, the identity map.
2. For each $t \in [0, 1)$, $\Gamma(t)$ is a homeomorphism of E onto E and $\Gamma^{-1}(t) \in C(E \times [0, 1), E)$.
3. $\Gamma(1)E$ is a single point in E and $\Gamma(t)A$ converges uniformly to $\Gamma(1)E$ as $t \to 1$ for each bounded set $A \subset E$.
4. For each $t_0 \in [0, 1)$ and each bounded set $A \subset E$,

$$\sup_{\substack{0\le t\le t_0\\ u\in A}} \{\|\Gamma(t)u\| + \|\Gamma^{-1}(t)u\|\} < \infty. \tag{3.1}$$

We have

Theorem 3.1. *Let G be a C^1-functional on E, and let A be a subset of E. Assume that*

$$a_0 := \sup_A G < a := \inf_{\Gamma\in\Phi} \sup_{\substack{0\le s\le 1\\ u\in A}} G(\Gamma(s)u) < \infty. \tag{3.2}$$

Let $\psi(t)$ be a positive, nonincreasing, locally Lipschitz continuous function on $[0, \infty)$ satisfying (2.8). Then there is a sequence $\{u_k\} \subset E$ such that (2.9) holds. In particular, there is a Cerami sequence satisfying (1.5).

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Proof. In this case, we let $\mathcal{K}$ be the collection of sets of the form

$$K = \{\Gamma(t)u : t \in [0,1],\ u \in A,\ \Gamma \in \Phi\}. \tag{3.3}$$

To show that this is a minimax system, let φ be a mapping in $\Lambda(A)$, and let $\Gamma(t)$ be in Φ. Define the mapping

$$\Gamma_1(t)u = \varphi \circ \Gamma(t) \circ \varphi^{-1}u, \quad u \in E.$$

Then $\Gamma_1(t) \in \Phi$. Moreover,

$$\Gamma_1(t)u = \varphi \circ \Gamma(t)u, \quad u \in A.$$

Consequently,

$$\Gamma_1(t)A = \varphi(\{\Gamma(t)u : u \in A\}),$$

showing that the collection $\mathcal{K}$ is a minimax system for A. Since (3.2) now becomes (2.7), the result follows. □

Definition 3.2. *We say that A links B [hm] if A, B are subsets of E such that $A \cap B = \phi$ and, for each $\Gamma(t) \in \Phi$, there is a $t \in (0,1]$ such that $\Gamma(t)A \cap B \neq \phi$.*

We have

Corollary 3.3. *If A links B [hm], then it links B [mm].*

We also have

Theorem 3.4. *Let A, B be subsets of E such that A links B [hm], and let G be a C^1-functional on E, satisfying*

$$a_0 := \sup_A G \le b_0 := \inf_B G. \tag{3.4}$$

Assume that

$$a := \inf_{\Gamma \in \Phi} \sup_{\substack{0 \le s \le 1 \\ u \in A}} G(\Gamma(s)u) \tag{3.5}$$

is finite. Let $\psi(t)$ be a positive, nonincreasing, locally Lipschitz continuous function on $[0,\infty)$ satisfying (2.8). Then there is a sequence $\{u_k\} \subset E$ such that (2.9) holds. In particular, there is a Cerami sequence satisfying (1.5).

Proof. The theorem follows from Theorem 2.12. If

$$K = \{\Gamma(t)u : t \in [0,1],\ u \in A\} \tag{3.6}$$

for some $\Gamma \in \Phi$, and $\sigma(t)$ is any flow, let

$$\tilde{\Gamma}(s) = \begin{cases} \sigma(2Ts), & 0 \le s \le \frac{1}{2}, \\ \sigma(T)\Gamma(2s-1), & \frac{1}{2} < s \le 1. \end{cases} \tag{3.7}$$

It is easily checked that $\tilde{\Gamma} \in \Phi$. Moreover,

$$\tilde{K} \subset \bigcup_{t \in [0,T]} \sigma(t)A \ \cup\ \sigma(T)K. \tag{3.8}$$

Hence, (2.14) is satisfied. □

3.3 A method using metric spaces

Another general procedure is described in Mawhin–Willem [96] and Brezis–Nirenberg [28] as follows. One finds a compact metric space Σ and selects a closed subset Σ^* of Σ such that $\Sigma^* \neq \phi,\ \ \Sigma^* \neq \Sigma$. One then picks a map $p^* \in C(\Sigma, E)$ and defines

$$\begin{aligned} \mathcal{A} =& \{p \in C(\Sigma, E) :\ p = p^* \text{ on } \Sigma^*\}, \\ a =& \inf_{p \in \mathcal{A}} \max_{\xi \in \Sigma} G(p(\xi)). \end{aligned}$$

They assume

(A) For each $p \in \mathcal{A}$, $\max_{\xi \in \Sigma} G(p(\xi))$ is attained at a point in $\Sigma \backslash \Sigma^*$.

They then prove

Theorem 3.5. *Under the above hypotheses, there is a sequence satisfying (1.4).*

We shall prove

Theorem 3.6. *Under the same hypotheses, let $\psi(t)$ be a positive, nonincreasing, locally Lipschitz continuous function on $[0, \infty)$ satisfying (2.8). Then there is a sequence $\{u_k\} \subset E$ such that (2.9) holds. In particular, there is a Cerami sequence satisfying (1.5).*

Proof. If $a > a_0$, this follows from Theorem 2.4. In fact, we can take $A = p^*(\Sigma^*)$ and $\mathcal{K}$ as the collection of sets

$$K = \{p(\xi) :\ \xi \in \Sigma,\ \ p \in \mathcal{A}\}.$$

If $\varphi \in \Lambda(A)$, then

$$\varphi(p(\xi)) = \varphi(p^*(\xi)) = p^*(\xi), \quad \xi \in \Sigma^*.$$

Thus, $\varphi(p(\xi)) \in \mathcal{K}$. Hence, $\mathcal{K}$ is a minimax system for A, and we can apply Theorem 2.4 to come to the desired conclusion.

Now assume that $a_0 = a$. Since each $K \in \mathcal{K}$ is compact, the same is true of K_ρ given by (2.25) for each $\rho > 0$. Moreover, if $\sigma(t) \in C(\mathbb{R}^+ \times E, E)$ is such that $\sigma(0) = I$, one has $S(K) \in \mathcal{K}$, where

$$S(p(\xi)) = \sigma(d(p(\xi), A))p(\xi), \quad \xi \in \Sigma.$$

The result follows from Theorem 2.14. $\square$

3.4 A method using homotopy-stable families

Another approach is found in Ghoussoub [74]. The author considers a closed subset A of E, and a collection $\mathcal{K}$ of compact subsets of E having the property that if $K \in \mathcal{K}$ and $\psi \in C([0, 1] \times E, E)$ satisfies

$$\psi(0)u = u, \quad u \in E,$$

and

$$\psi(t)u = u, \quad t \in [0, 1], \ u \in A,$$

then $\psi(1)K \in \mathcal{K}$. He calls such a collection *a homotopy-stable family with extended boundary A*. He proves

Theorem 3.7. *If $\mathcal{K}$ is a homotopy-stable family with extended closed boundary A, B is a closed subset of E satisfying (1.1), and G is a C^1-functional that satisfies*

$$\sup_A G \le a \le \inf_B G,$$

where the quantity a is given by (2.6), then there is a PS sequence satisfying (1.4).

We shall prove

Theorem 3.8. *Under the same assumptions, for each function $\psi(t)$ satisfying the hypotheses of Theorem 2.4, there is a sequence satisfying (2.9). In particular, there is a Cerami sequence satisfying (1.5).*

Proof. It is easy to show that such a collection $\mathcal{K}$ is indeed a minimax system. Indeed, if $K \in \mathcal{K}$ and $\varphi \in \Lambda(A)$, then

$$\psi(t)u = t\varphi(u) + (1 - t)u$$

satisfies the stipulations above, and consequently $\varphi(K) \in \mathcal{K}$. Since each of the members of $\mathcal{K}$ is compact, it follows that every C^1-functional is Lipschitz continuous on some set K_ρ defined by (2.25) for $\rho > 0$ sufficiently small. Moreover, if $\sigma(t) \in C(\mathbb{R}^+ \times E, E)$ is such that $\sigma(0) = I$, one has $S(K) \in \mathcal{K}$, where

$$S(u) = \sigma(d(u, A))u, \quad u \in E.$$

This follows from the fact that $S(t)u = \sigma(td(u, A))u$ is in $\in C([0, 1] \times E, E)$ and satisfies

$$S(0)u = u, \quad u \in E,$$

and

$$S(t)u = u, \quad t \in [0, 1], \ u \in A.$$

Consequently, $S(K) = S(1)K \in \mathcal{K}$ by hypothesis. It now follows that the conclusion of Theorem 2.14 holds. □

Note. It is not required to have a satisfy

$$a \le \inf_B G.$$

If it does, one obtains additional information as described in [74].

3.5 Examples of linking sets

We now discuss various subsets of a Banach space E with respect to linking. First we have

Proposition 3.9. *[122] Let A, B be two closed, bounded subsets of E such that $E \backslash A$ is path connected. If A links B [hm], then B links A [hm].*

The next proposition gives a very useful method of checking the linking of two sets.

Proposition 3.10. *[122] Let F be a continuous map from E to $\mathbb{R}^n$, and let $Q \subset E$ be such that $F_0 = F|_Q$ is a homeomorphism of Q onto the closure of a bounded open subset Ω of $\mathbb{R}^n$. If $p \in \Omega$, then $F_0^{-1}(\partial\Omega)$ links $F^{-1}(p)$ [hm].*

Proposition 3.11. *[122] If H is a homeomorphism of E onto itself and A links B [hm], then HA links HB [hm].*

The following examples were given in [122].

Example 1. Let B be an open set in E, and let A consist of two points e_1, e_2 with $e_1 \in B$ and $e_2 \notin \bar{B}$. Then A links ∂B [hm]. ∂B links A [hm] as well if ∂B is bounded.

Example 2. Let M, N be closed subspaces such that $\dim N < \infty$ and $E = M \oplus N$. Let

$$B_R = \{u \in E : \|u\| < R\} \tag{3.9}$$

and take $A = \partial B_R \cap N,\ \ B = M$. Then A links B [hm].

Example 3. We take M, N as in Example 2. Let $w_0 \neq 0$ be an element of M, and take

$$\begin{aligned} A &= \{v \in N : \|v\| \leq R\} \cup \{sw_0 + v : v \in N, s \geq 0, \|sw_0 + v\| = R\}, \\ B &= \partial B_\delta \cap M,\ \ 0 < \delta < R. \end{aligned}$$

Then A and B link each other [hm].

Example 4. Take M, N as before and let $v_0 \neq 0$ be an element of N. We write $N = \{v_0\} \oplus N'$. We take

$$\begin{aligned} A &= \{v' \in N' : \|v'\| \leq R\} \cup \{sv_0 + v' : v' \in N', s \geq 0, \|sv_0 + v'\| = R\}, \\ B &= \{w \in M : \|w\| \geq \delta\} \cup \{sv_0 + w : w \in M, s \geq 0, \|sv_0 + w\| = \delta\}, \end{aligned}$$

where $0 < \delta < R$. Then A links B [hm].

Example 5. This is the same as Example 4 with A replaced by $A = \partial B_R \ \cap\ N$.

Example 6. Let M, N be as in Example 2. Take $A = \partial B_\delta \cap N$, and let v_0 be any element in $\partial B_1 \cap N$. Take B to be the set of all u of the form

$$u = w + sv_0,\ \ w \in M,$$

satisfying any of the following:

(a) $\|w\| \leq R,\ s = 0$

(b) $\|w\| \leq R,\ s = 2R_0$

(c) $\|w\| = R,\ 0 \leq s \leq 2R_0,$

where $0 < \delta < \min(R, R_0)$. Then A and B link each other [hm].

Example 7. Let M, N be as in Example 2. Let v_0 be in $\partial B_1 \cap N$ and write $N = \{v_0\} \oplus N'$. Let

$$A = \partial B_\delta \cap N,\ \ Q = \bar{B}_\delta \cap N,$$

and

$$B = \{w \in M : \|w\| \leq R\} \cup \{w + sv_0 : w \in M, s \geq 0, \|w + sv_0\| = R\},$$

where $0 < \delta < R$. Then A and B link each other [hm].

Example 8. Let M, N be closed subspaces of E, one of which is finite-dimensional, and such that

$$E = M \oplus N.$$

If

$$B_R := \{u \in E : \|u\| < R\},$$

then $M \cap \partial B_R$ links N [hm] for each $R > 0$.

Example 9. Let M, N be closed subspaces of E such that

$$E = M \oplus N,$$

with one of them being finite-dimensional. Let w_0 be an element of $M \backslash \{0\}$, and let $0 < \delta < r < R$. Take

$$\begin{aligned} A &= \{v \in N : \delta \leq \|v\| \leq R\} \cup \{sw_0 + v : v \in N, s \geq 0, \|sw_0 + v\| = \delta\} \\ &\cup \{sw_0 + v : v \in N, s \geq 0, \|sw_0 + v\| = R\}, \\ \\ B &= \partial B_r \cap M,\ 0 < \delta < r < R. \end{aligned}$$

Then A and B link each other [hm].

Example 10. Let M, N be closed subspaces of E such that

$$E = M \oplus N,$$

with one of them being finite-dimensional. Let w_0 be an element of $M \backslash \{0\}$, and let $0 < r < R$,

$$\begin{aligned} A &= \{w \in M : \|w\| = R\}, \\ B &= \{v \in N : \|v\| \geq r\} \cup \{u = v + sw_0 : v \in N, s \geq 0, \|u\| = r\}. \end{aligned}$$

Then A links B [hm].

Example 11. Let M, N be as in Example 2. Take $A = \partial B_\delta \cap N$, and let v_0 be any element in $\partial B_1 \cap N$. Take B to be the set of all u of the form

$$u = w + sv_0, \quad w \in M,$$

satisfying any of the following:

(a) $s = 0$

(b) $s = 2R_0$

where $0 < \delta < R_0$. Then A links B [hm].

Example 12. Let M, N be as in Example 2. Take $A = \partial B_\delta \cap N$, and let v_0 be any element in $\partial B_1 \cap N$. Take B to be the set of all u of the form

$$u = w + sv_0, \quad w \in M,$$

satisfying any of the following:

(a) $\|w\| \le R, \ s = 0,$

(b) $\|w\| = R, \ s > 0,$

where $0 < \delta < \infty$. Then A links B [hm].

Example 13. Let M be a closed subspace of a Hilbert space E with complement $N \oplus \{v_0\}$, where v_0 is an element in E having unit norm, and let δ be any positive number. Let $\varphi(t) \in C^1(\mathbb{R})$ be such that

$$0 \le \varphi(t) \le 1, \quad \varphi(0) = 1,$$

and

$$\varphi(t) = 0, \quad |t| \ge 1.$$

Let

$$F(v + w + sv_0) = w + [s + \delta - \delta\varphi(\|v\|^2/\delta^2)]v_0, \quad v \in N, \ w \in M, \ s \in \mathbb{R}.$$

Assume that one of the subspaces M, N is finite-dimensional. Take

$$A = [M \oplus \{v_0\}] \cap \partial B_R$$

and

$$B = \{v + rv_0 : v \in N, r = \delta\varphi(\|v\|^2/\delta^2)\}.$$

Then A links B [hm] provided $0 < \delta < R$.

Proposition 3.12. *[122] If K is any subset of a bounded open set $\Omega \subset E$, then $\partial\Omega$ links K [hm].*

3.6 Various geometries

We now apply the theorems of the preceding sections to various geometries in Banach space. As before, we assume that $G \in C^1(E, \mathbb{R})$ and that ψ satisfies the hypotheses of Theorem 2.4.

Theorem 3.13. *Assume that there is a $\delta > 0$ such that*

$$G(0) \leq \alpha \leq G(u), \quad u \in \partial B_\delta, \tag{3.10}$$

and that there are a $R_0 < \infty$ and a $\varphi_0 \in \partial B_1$ such that

$$G(R\varphi_0) \leq \gamma, \quad R > R_0. \tag{3.11}$$

Then, for each function $\psi(t)$ satisfying the hypotheses of Theorem 2.4, there is a sequence $\{u_k\} \subset E$ such that

$$G(u_k) \to c, \quad \alpha \leq c \leq \gamma, \quad G'(u_k)/\psi(\|u_k\|) \to 0. \tag{3.12}$$

Proof. We take $A = \{0, R\varphi_0\}$, $B = \partial B_\delta$. Then $A'' = \{R\varphi_0\}$. Note that a given by (2.6) is finite for each R since

$$a_R \leq \max_{0 \leq r \leq R} G(r\varphi_0).$$

We apply Theorem 2.24. We note that in each case

$$a_R \leq \gamma, \quad R > 0, \tag{3.13}$$

since the mapping

$$\Gamma(s)u = (1-s)u \tag{3.14}$$

(which is in Φ) satisfies

$$G(\Gamma(s)u) \leq \gamma, \quad 0 \leq s \leq 1, \ u \in A. \tag{3.15}$$

This implies (3.13). We replace $\psi(t)$ with $\tilde{\psi}(t) = \psi(t + \delta)$, which also satisfies the hypotheses of Theorem 2.4. By Theorem 2.24, we can find a sequence satisfying

$$\alpha - (1/k) \leq G(u_k) \leq \gamma + (1/k), \quad G'(u_k)/\tilde{\psi}(d(u_k, B)) \to 0. \tag{3.16}$$

This implies (3.12), since $\|u\| \leq d(u, B) + \delta$. □

Theorem 3.14. *Let M, N be closed subspaces of E such that*

$$E = M \oplus N, \quad M \neq E, \quad N \neq E, \tag{3.17}$$

with

$$\dim M < \infty \ \textit{ or } \ \dim N < \infty. \tag{3.18}$$

Let $G \in C^1(E, \mathbb{R})$ be such that

$$G(v) \le \gamma, \quad v \in \partial B_R \cap N, \; R > R_0, \tag{3.19}$$

and

$$G(w) \ge \alpha, \quad w \in M. \tag{3.20}$$

Then, for each function $\psi(t)$ satisfying the hypotheses of Theorem 2.4, there is a sequence $\{u_k\} \subset E$ such that

$$G(u_k) \to c, \quad \alpha \le c \le \gamma, \quad G'(u_k)/\psi(d(u_k, M)) \to 0. \tag{3.21}$$

Proof. This time we take A and B as in Example 2 above. Thus, A links B [hm]. Again, a_R given by (2.6) is finite for each R since

$$a_R \le \max_{u \in \bar{B}_R \cap N} G(u).$$

Again we see that we can apply Theorem 2.24 to conclude that the desired sequence exists. □

Theorem 3.15. *Let M, N be as in Theorem 3.14, and let $G \in C^1(E, \mathbb{R})$ satisfy*

$$G(v) \le \alpha, \quad v \in N, \tag{3.22}$$

$$G(w) \ge \alpha, \quad w \in \partial B_\delta \cap M, \tag{3.23}$$

$$G(sw_0 + v) \le \gamma, \quad s \ge 0, \; v \in N, \; \|sw_0 + v\| = R > R_0, \tag{3.24}$$

for some $w_0 \in \partial B_1 \cap M$, where $0 < \delta < R_0$. Then, for each function $\psi(t)$ satisfying the hypotheses of Theorem 2.4, there is a sequence $\{u_k\} \subset E$ such that (3.12) holds.

Proof. Here we take A, B as in Example 3 above. Thus, A and B link each other [hm]. Here

$$A'' = \{sw_0 + v : s \ge 0, v \in N, \|sw_0 + v\| = R\}.$$

Again, for each R, the quantity a given by (2.6) is finite since

$$a_R \le \max_Q G,$$

where

$$Q = \{sw_0 + v : s \ge 0, v \in N, \|sw_0 + v\| \le R\}.$$

We now apply Theorem 2.24 to conclude that the desired sequence exists. □

Theorem 3.16. *Let M, N be as in Theorem 3.14, and let $v_0 \in \partial B_1 \cap N$. Take $N = \{v_0\} \oplus N'$. Let $G \in C^1(E, \mathbb{R})$ be such that*

$$G(v) \le \gamma, \quad v \in \partial B_R \cap N, \; R > R_0, \tag{3.25}$$

$$G(w) \ge \alpha, \quad w \in M, \; \|w\| \ge \delta, \tag{3.26}$$

$$G(sv_0 + w) \ge \alpha, \quad s \ge 0, \; w \in M, \; \|sv_0 + w\| = \delta, \tag{3.27}$$

where $0 < \delta < R_0$. Then, for each function $\psi(t)$ satisfying the hypotheses of Theorem 2.4, there is a sequence $\{u_k\} \subset E$ such that (3.12) holds.

Proof. We take A, B as in Example 5 above. Thus, A links B [hm]. As before, we note that $a_R < \infty$ for each R. Hence, (3.12) holds by Theorem 2.24. $\square$

3.7 A sandwich theorem

We now discuss a very useful theorem that allows one to consider functionals that are bounded from below on one subspace and bounded from above on another with no correlation between the bounds. This provides such functionals with the same advantages as those that are semibounded. One drawback is the requirement that one of the subspaces be finite-dimensional. This condition will be removed in Chapter 15 if we assume that the functional satisfies more than the mere continuity of its derivative.

Theorem 3.17. *Let N be a closed subspace of a Hilbert space E and let $M = N^{\perp}$. Assume that at least one of the subspaces M, N is finite-dimensional. Let G be a C^1-functional on E such that*

$$m_0 := \inf_{w \in M} G(w) \neq -\infty \tag{3.28}$$

and

$$m_1 := \sup_{v \in N} G(v) \neq \infty. \tag{3.29}$$

Let $\psi(t)$ be a positive, nonincreasing, locally Lipschitz continuous function on $[0, \infty)$ such that (2.8) holds. Then there are a constant $c \in \mathbb{R}$ and a sequence $\{u_k\} \subset E$ such that

$$G(u_k) \to c, \quad m_0 \le c \le m_1, \quad G'(u_k)/\psi(\|Pu_k\|) \to 0, \tag{3.30}$$

where P is the (orthogonal) projection onto M.

Proof. We may assume $\dim N < \infty$; otherwise, we can consider $-G$ in place of G. Let A be the set $\partial B_R \cap N$, and take $B = M$, where $R > 0$ is arbitrary. Then A links B [hm] by Example 2 above. We now apply Theorem 2.21. Note that $a_0 \le m_1,\ m_0 = b_0$, and (2.31) holds for R sufficiently large. We also note that

$$a_R \le \sup_{B_R \cap N} G \le m_1$$

by taking $\Gamma(s)u = (1 - s)u,\ u \in E$. Hence, by Theorem 2.21, for each $\delta > 0$, there is a $u \in E$ such that

$$m_0 - \delta \le G(u) \le m_1 + \delta, \quad \|G'(u)\| < \psi(d(u, B')).$$

Since this is true for each $\delta > 0$, we obtain the desired conclusion. $\square$

An immediate consequence is

Corollary 3.18. *Under the hypotheses of Theorem 3.17, there are a constant $c \in \mathbb{R}$ and a sequence $\{u_k\} \subset E$ such that*

$$G(u_k) \to c, \quad m_0 \le c \le m_1, \quad (1 + \|Pu_k\|)G'(u_k) \to 0, \tag{3.31}$$

where P is the (orthogonal) projection onto M.

The following is a consequence of Theorem 2.23.

Theorem 3.19. *Under the hypotheses of Theorem 3.17, for any sequence $\{R_k\} \subset \mathbb{R}^+$ such that $R_k \to \infty$, there are a constant $c \in \mathbb{R}$ and a sequence $\{u_k\} \subset E$ such that*

$$G(u_k) \to c, \quad m_0 \le c \le m_1, \quad (R_k + \|u_k\|)\|G'(u_k)\| \le \frac{m_1 - m_0}{\ln(4/3)}. \tag{3.32}$$

Proof. We may assume $\dim N < \infty$. Let A_k be the set $\partial B_{R_k} \cap N$, and take $B_k = M$. Then, for each k, A_k links B_k [hm] by Example 2 above. We now apply Theorem 2.23. Note that $\alpha_k = R_k$ and $a_{k0} \le m_1, m_0 = b_{k0}$. Take

$$\psi_k(t) = \frac{m_1 - m_0}{[2R_k + t]\ln(4/3)}.$$

Since $R_k + d(u, B_k') \ge \|u\|$, we see that (3.32) holds for each k. □

The following is another consequence of Theorem 2.23.

Theorem 3.20. *Let N be a closed subspace of a Hilbert space E, and let $M = N^{\perp}$. Assume that at least one of the subspaces M, N is finite-dimensional. Let G be a C^1-functional on E such that*

$$m_0 := \sup_{v \in N} \inf_{w \in M} G(v + w) \ne -\infty \tag{3.33}$$

and

$$m_1 := \inf_{w \in M} \sup_{v \in N} G(v + w) \ne \infty. \tag{3.34}$$

Then, for any $\varepsilon > 0$ and any sequence $\{R_k\} \subset \mathbb{R}^+$ such that $R_k \to \infty$, there are a constant $c \in \mathbb{R}$ and a sequence $\{u_k\} \subset E$ such that

$$\begin{gathered} G(u_k) \to c, \quad m_0 - \varepsilon \le c \le m_1 + \varepsilon, \\ (R_k + \|u_k\|)\|G'(u_k)\| \le \frac{m_1 - m_0 + 3\varepsilon}{\ln(5/4)}. \end{gathered} \tag{3.35}$$

Proof. We may assume $\dim N < \infty$. Let $\varepsilon > 0$ be given. Then there is a $u_\varepsilon = v_\varepsilon + w_\varepsilon$ such that

$$m_0 - \varepsilon < \inf_{w \in M} G(v_\varepsilon + w), \quad \sup_{v \in N} G(v + w_\varepsilon) < m_1 + \varepsilon.$$

Note that $\alpha_k = R_k$ and $a_{k0} \le m_1, m_0 = b_{k0}$. Take

$$\psi_k(t) = \frac{m_1 - m_0 + 3\varepsilon}{[3R_k + t]\ln(5/4)}.$$

Note that

$$m_1 - m_0 + 2\varepsilon < \int_{R_k}^{2R_k} \psi_k(t)\,dt.$$

We now apply Theorem 2.23. Thus, there is a sequence such that

$$G(u_k + u_\varepsilon) \to c, \quad m_0 - \varepsilon \le c \le m_1 + \varepsilon,$$

and

$$[3R_k + d(u_k, B_k')]\|G'(u_k + u_\varepsilon)\| \le \frac{m_1 - m_0 + 3\varepsilon}{\ln(5/4)}.$$

Since $R_k + d(u, B_k') \ge \|u\|$, we have

$$2R_k + d(u_k, B_k') \ge R_k + \|u_k\| \ge \|u_\varepsilon\| + \|u_k\| \ge \|u_\varepsilon + u_k\|$$

when $R_k \ge \|u_\varepsilon\|$. Let $h_k = u_k + u_\varepsilon$. Then we have

$$G(h_k) \to c, \quad m_0 - \varepsilon \le c \le m_1 + \varepsilon,$$
$$(R_k + \|h_k\|)\|G'(h_k)\| \le \frac{m_1 - m_0 + 3\varepsilon}{\ln(5/4)}.$$

Since ε was arbitrary, we see that (3.35) holds. □

Here are some consequences.

Theorem 3.21. *Let G be a C^1-functional on E such that*

$$a_0 = \sup_E G < \infty. \tag{3.36}$$

If ψ satisfies the hypotheses of Theorem 2.4, then there is a sequence $\{u_k\} \subset E$ such that

$$G(u_k) \to a_0, \quad G'(u_k)/\psi(\|u_k\|) \to 0. \tag{3.37}$$

The same holds if

$$b_0 = \inf_E G > -\infty, \tag{3.38}$$

with

$$G(u_k) \to b_0, \quad G'(u_k)/\psi(\|u_k\|) \to 0. \tag{3.39}$$

Proof. We refer to Theorem 2.24. We take a sequence of points such that $G(v_k) > a_0 - (1/k)$ and a sequence $\{R_k\}$ such that $R_k > 2\|v_k\|$ and

$$c_k = 2\left[k\int_{2\beta_k}^{R_k+2\beta_k} \psi(t)\,dt\right]^{-1} \to 0, \quad k \to \infty,$$

where $\beta_k = \|v_k\|$. We then take

$$\psi_k(t) = c_k\psi(t+\beta_k)$$

in Theorem 2.24. Then (2.53) holds. We used the fact that $\partial B_{R_k+\beta_k}$ links $\{v_k\}$ for each k (Theorem 3.12). The conclusion follows since

$$\|u\| \leq d(u, v_k) + \beta_k.$$

In the second case, we replace G with $-G$. $\square$

Corollary 3.22. *If (3.36) holds, then there is a sequence satisfying*

$$G(u_k) \to a_0, \quad (1 + \|u_k\|)G'(u_k) \to 0. \tag{3.40}$$

If (3.38) holds, there is a sequence satisfying

$$G(u_k) \to b_0, \quad (1 + \|u_k\|)G'(u_k) \to 0. \tag{3.41}$$

3.8 Notes and remarks

The results of Section 3.2 are from [136], [114], and [120] (cf. also [122]). Sections 3.5 and 3.6 are from [122]. The results of Section 3.7 come from [143], [109], [108], [129], and [132].

Chapter 4

Ordinary Differential Equations

4.1 Extensions of Picard's theorem

In proving the theorems of Chapter 2, we shall make use of various extensions of Picard's theorem in a Banach space. Some are well known, and some are of interest in their own right. All of them will be used in proving the theorems of Chapter 2.

Theorem 4.1. *Let X be a Banach space, and let*

$$B_0 = \{x \in X : \|x - x_0\| \leq R_0\}$$

and

$$I_0 = \{t \in \mathbb{R} : |t - t_0| \leq T_0\}.$$

Assume that $g(t, x)$ is a continuous map of $I_0 \times B_0$ into X such that

$$\|g(t, x) - g(t, y)\| \leq K_0\|x - y\|, \quad x, y \in B_0,\ t \in I_0, \tag{4.1}$$

and

$$\|g(t, x)\| \leq M_0, \quad x \in B_0,\ t \in I_0. \tag{4.2}$$

Let T_1 be such that

$$T_1 \leq \min(T_0,\ R_0/M_0), \quad K_0 T_1 < 1. \tag{4.3}$$

Then there is a unique solution $x(t)$ of

$$\frac{dx(t)}{dt} = g(t, x(t)), \quad |t - t_0| \leq T_1, \quad x(t_0) = x_0. \tag{4.4}$$

Lemma 4.2. *Let $\gamma(t)$ and $\rho(t)$ be continuous functions on $[0, \infty)$, with $\gamma(t)$ nonnegative and $\rho(t)$ positive and nondecreasing. Assume that*

$$\int_{u_0}^{\infty} \frac{d\tau}{\rho(\tau)} > \int_{t_0}^{T} \gamma(s)\, ds, \tag{4.5}$$

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where $t_0 < T$ *and* $u_0 \geq 0$ *are given numbers. Then there is a unique solution of*

$$u'(t) = \gamma(t)\rho(u(t)), \quad t \in [t_0, T), \; u(t_0) = u_0 \tag{4.6}$$

that is positive in $[t_0, T)$ *and depends continuously on* u_0.

Proof. One can separate variables to obtain

$$W(u) \equiv \int_{u_0}^{u} \frac{d\tau}{\rho(\tau)} = \int_{t_0}^{t} \gamma(s)\, ds.$$

The function $W(u)$ is differentiable and increasing in $\mathbb{R}$, is positive in (u_0, ∞), depends continuously on u_0, and satisfies

$$W(u) \to L = \int_{u_0}^{\infty} \frac{d\tau}{\rho(\tau)} > \int_{t_0}^{T} \gamma(s)\, ds \quad \text{as } u \to \infty.$$

Thus, for each $t \in [t_0, T)$, there is a unique $u \in [u_0, \infty)$ such that

$$W(u) = \int_{t_0}^{t} \gamma(s)\, ds.$$

Hence,

$$u = W^{-1}\left(\int_{t_0}^{t} \gamma(s)\, ds\right)$$

is the unique solution of (4.6) and depends continuously on u_0. □

Lemma 4.3. *Let* $\gamma(t)$ *and* $\rho(t)$ *be continuous functions on* $[0, \infty)$*, with* $\gamma(t)$ *nonnegative and* $\rho(t)$ *positive and nondecreasing. Assume that*

$$\int_{m}^{u_0} \frac{d\tau}{\rho(\tau)} > \int_{t_0}^{T} \gamma(s)\, ds, \tag{4.7}$$

where $t_0 < T$ *and* $m < u_0$ *are given numbers. Then there is a unique solution of*

$$u'(t) = -\gamma(t)\rho(u(t)), \quad t \in [t_0, T), \; u(t_0) = u_0, \tag{4.8}$$

which is $\geq m$ *in* $[t_0, T)$ *and depends continuously on* u_0.

Proof. One can separate variables to obtain

$$W(u) \equiv \int_{u}^{u_0} \frac{d\tau}{\rho(\tau)} = \int_{t_0}^{t} \gamma(s)\, ds.$$

The function $W(u)$ is differentiable and decreasing in $\mathbb{R}$, is positive in $[m, u_0]$, depends continuously on u_0, and satisfies

$$W(u) \to L = \int_{m}^{u_0} \frac{d\tau}{\rho(\tau)} > \int_{t_0}^{T} \gamma(s)\, ds \quad \text{as } u \to m.$$

Thus, for each $t \in [t_0, T)$, there is a unique $u \in [m, u_0]$ such that

$$W(u) = \int_{t_0}^{t} \gamma(s)\, ds.$$

Hence,

$$u = W^{-1}\left(\int_{t_0}^{t} \gamma(s)\, ds\right)$$

is the unique solution of (4.8) and depends continuously on u_0. $\square$

4.2 Estimating solutions

Theorem 4.4. *Assume, in addition to the hypotheses of Theorem 4.1, that*

$$\|g(t,x)\| \le \gamma(t)\rho(\|x\|), \quad x \in B_0,\ t \in I_0, \tag{4.9}$$

where $\gamma(t)$ and $\rho(t)$ satisfy the hypotheses of Lemma 4.2 with $T = t_0 + T_1$. Let $u(t)$ be the positive solution of

$$u'(t) = \gamma(t)\rho(u(t)),\quad t \in [t_0, T),\quad u(t_0) = u_0 \ge \|x_0\| \tag{4.10}$$

provided by that lemma. Then the unique solution of (4.4) satisfies

$$\|x(t)\| \le u(t), \quad t \in [t_0, T). \tag{4.11}$$

Proof. Assume that there is a $t_1 \in [t_0, T)$ such that

$$u(t_1) < \|x(t_1)\|.$$

For $\varepsilon > 0$, let $u_\varepsilon(t)$ be the solution of

$$u'(t) = [\gamma(t) + \varepsilon]\rho(u(t)), \quad t \in [t_0, T),\ u(t_0) = u_0. \tag{4.12}$$

By Lemma 4.2, a solution exists for $\varepsilon > 0$ sufficiently small. Moreover, $u_\varepsilon(t) \to u(t)$ uniformly on any compact subset of $[t_0, T)$. Let

$$w(t) = \|x(t)\| - u_\varepsilon(t).$$

Then we may take ε sufficiently small so that

$$w(t_0) \le 0, \quad w(t_1) > 0.$$

Let t_2 be the largest number in $[t_0, t_1)$ such that $w(t_2) = 0$ and

$$w(t) > 0, \quad t \in (t_2, t_1].$$

For $h > 0$ sufficiently small, we have

$$\frac{w(t_2 + h) - w(t_2)}{h} > 0.$$

Consequently,

$$D^+ w(t_2) \geq 0.$$

But

$$(4.13)\qquad \begin{aligned} D^+ w(t_2) &= D^+\|x(t_2)\| - u'_\varepsilon(t_2) \\ &\leq \|x'(t_2)\| - u'_\varepsilon(t_2) \\ &= \|g(t_2, x(t_2))\| - [\gamma(t_2) + \varepsilon]\rho(u_\varepsilon(t_2)) \\ &\leq \gamma(t_2)\rho(\|x(t_2)\|) - [\gamma(t_2) + \varepsilon]\rho(u_\varepsilon(t_2)) \\ &= -\varepsilon\rho(u_\varepsilon(t_2)) \\ &< 0. \end{aligned}$$

This contradiction proves the theorem. □

4.3 Extending solutions

Theorem 4.5. *Let $g(t, x)$ be a continuous map from $\mathbb{R} \times H$ to H, where H is a Banach space. Assume that for each point $(t_0, x_0) \in \mathbb{R} \times H$, there are constants $K, b > 0$ such that*

$$(4.14)\quad \|g(t,x) - g(t,y)\| \leq K\|x - y\|, \quad |t - t_0| < b,\ \|x - x_0\| < b,\ \|y - x_0\| < b.$$

Assume also that

$$(4.15)\qquad \|g(t,x)\| \leq \gamma(t)\rho(\|x\|), \quad x \in H,\ t \in [t_0, T_M),$$

where $T_M \leq \infty$, and $\gamma(t)$, $\rho(t)$ satisfy the hypotheses of Lemma 4.2. Then, for each $x_0 \in H$ and $t_0 > 0$, there is a unique solution $x(t)$ of the equation

$$(4.16)\qquad \frac{dx(t)}{dt} = g(t, x(t)), \quad t \in [t_0, T_M),\ x(t_0) = x_0.$$

Moreover, $x(t)$ depends continuously on x_0 and satisfies

$$(4.17)\qquad \|x(t)\| \leq u(t), \quad t \in [t_0, T_M),$$

where $u(t)$ is the solution of (4.6) in that interval satisfying $u(t_0) = u_0 \geq \|x_0\|$.

Before proving Theorem 4.5, we note that the following is an immediate consequence.

Corollary 4.6. *Let $V(y)$ be a locally Lipschitz continuous map from H to itself satisfying*

$$(4.18)\qquad \|V(y)\| \leq C(1 + \|y\|), \quad y \in H.$$

Then, for each $y_0 \in H$, there is a unique solution of

$$(4.19)\qquad y'(t) = V(y(t)), \quad t \in \mathbb{R}^+,\ y(0) = y_0.$$

4.4 The proofs

We now give the proof of Theorem 4.5.

Proof. By Theorems 4.1 and 4.4, there is an interval $[t_0, t_0 + m], m > 0$, in which a unique solution of

$$\frac{dx(t)}{dt} = g(t, x(t)), \quad t \in [t_0, t_0 + m], \; x(t_0) = x_0 \tag{4.20}$$

exists and satisfies

$$\|x(t)\| \le u(t), \quad t \in [t_0, t_0 + m], \tag{4.21}$$

where $u(t)$ is the unique solution of

$$u'(t) = \gamma(t)\rho(u(t)), \quad t \in [t_0, T_M), \; u(t_0) = u_0 = \|x_0\|. \tag{4.22}$$

Let $T \le T_M$ be the supremum of all numbers $t_0 + m$ for which this holds. If $t_1 < t_2 < T$, then the solution in $[t_0, t_2]$ coincides with that in $[t_0, t_1]$, since such solutions are unique. Thus, a unique solution of (4.20) satisfying (4.21) exists for each $t_0 < t < T$. Moreover, we have

$$x(t_2) - x(t_1) = \int_{t_1}^{t_2} g(t, x(t))\, dt.$$

Consequently,

$$\begin{aligned}
\|x(t_2) - x(t_1)\| &\le \int_{t_1}^{t_2} \|g(t, x(t))\|\, dt \\
&\le \int_{t_1}^{t_2} \gamma(t)\rho(\|x(t)\|)\, dt \\
&\le \int_{t_1}^{t_2} \gamma(t)\rho(u(t))\, dt \\
&= u(t_2) - u(t_1).
\end{aligned}$$

Assume that $T < T_M$. Let t_k be a sequence such that $t_0 < t_k < T$ and $t_k \to T$. Then,

$$\|x(t_k) - x(t_j)\| \le u(t_k) - u(t_j) \to 0.$$

Thus, $\{x(t_k)\}$ is a Cauchy sequence in H. Since H is complete, $x(t_k)$ converges to an element $x_1 \in H$. Since $\|x(t_k)\| \le u(t_k)$, we see that $\|x_1\| \le u(T)$. Moreover, we note that

$$x(t) \to x_1 \text{ as } t \to T.$$

To see this, let $\varepsilon > 0$ be given. Then there is a k such that

$$\|x(t_k) - x_1\| < \varepsilon, \quad u(T) - u(t_k) < \varepsilon.$$

Then, for $t_k \le t < T$,

$$\begin{aligned}\|x(t) - x_1\| &\le \|x(t) - x(t_k)\| + \|x(t_k) - x_1\| \\ &\le u(t) - u(t_k) + \|x(t_k) - x_1\| < 2\varepsilon.\end{aligned}$$

We define $x(T) = x_1$. Then we have a solution of (4.20) satisfying (4.21) in $[0, T]$. By Theorem 4.1, there is a unique solution of

$$\frac{dy(t)}{dt} = g(t, y(t)), \quad y(T) = x_1 \tag{4.23}$$

satisfying $\|y(t)\| \le u(t)$ in some interval $|t - T| < \delta$. By uniqueness, the solution of (4.23) coincides with the solution of (4.20) in the interval $(T - \delta, T]$. Define

$$\begin{aligned} z(t) &= x(t), \quad t_0 \le t < T, \\ z(T) &= x_1, \\ z(t) &= y(t), \quad T < t \le T + \delta. \end{aligned}$$

This gives a solution of (4.20) satisfying (4.21) in the interval $[t_0, T + \delta)$, contradicting the definition of T. Hence, $T = T_M$. □

We also have the following.

Theorem 4.7. *Let $g(t, x)$ be a continuous map from $\mathbb{R} \times H$ to H, where H is a Banach space. Assume that for each point $(t_0, x_0) \in \mathbb{R} \times H$, there are constants $K, b > 0$ such that*

$$\|g(t, x) - g(t, y)\| \le K\|x - y\|, \quad |t - t_0| < b, \ \|x - x_0\| < b, \ \|y - x_0\| < b. \tag{4.24}$$

Assume also that $g(t, x)$ satisfies (4.15), where $T_M \le \infty$, and $\gamma(t)$, $\rho(t)$ satisfy the hypotheses of Lemma 4.2 with $\rho(t)$ nondecreasing. Then, for each $x_0 \in H$ and $t_0 \in \mathbb{R}$, there is a unique solution $x(t)$ of (4.16) depending continuously on x_0 and satisfying

$$\|x(t)\| \ge u(t), \quad t \in [t_0, T_M), \tag{4.25}$$

where $u(t)$ is the solution of (4.8) in that interval.

4.5 An important estimate

Lemma 4.8. *Let ρ, γ satisfy the hypotheses of Lemma 4.3, with ρ locally Lipschitz continuous. Let $u(t)$ be the solution of (4.8), and let $h(t)$ be a continuous function satisfying*

$$h(t) \ge h(s) - \int_s^t \gamma(r)\rho(h(r))\,dr, \quad t_0 \le s < t < T, \quad h(t_0) \ge u_0. \tag{4.26}$$

Then

$$u(t) \le h(t), \quad t \in [t_0, T). \tag{4.27}$$

Proof. Assume that there is a point t_1 in the interval such that

$$h(t_1) < u(t_1).$$

Let

$$y(t) = u(t) - h(t), \quad t \in [t_0, T).$$

Then $y(t_0) \leq 0$ and $y(t_1) > 0$. Let τ be the largest point $< t_1$ such that $y(\tau) = 0$. Then

$$y(t) > 0, \quad t \in (\tau, t_1]. \tag{4.28}$$

Moreover, by (4.8) and (4.26), we have

$$y(t) \leq -\int_\tau^t \gamma(s)[\rho(u(s)) - \rho(h(s))]\,ds \leq L\int_\tau^t y(s)\,ds, \tag{4.29}$$

where L is the Lipschitz constant for ρ at $u(\tau)$ times the maximum of γ in the interval. Let

$$w(t) = \int_\tau^t y(s)\,ds.$$

Then

$$[e^{-Lt}w(t)]' = e^{-Lt}[y(t) - Lw(t)] \leq 0, \quad t \in [\tau, t_1].$$

Consequently,

$$e^{-Lt}w(t) \leq e^{-L\tau}w(\tau) = 0, \quad t \in [\tau, t_1].$$

Hence,

$$y(t) \leq Lw(t) \leq 0, \quad t \in [\tau, t_1],$$

contradicting (4.28). This completes the proof. □

Chapter 5

The Method Using Flows

5.1 Introduction

In this chapter we give the proofs of the theorems of Chapter 2. They rely on the theorems for ordinary differential equations in abstract spaces developed in Chapter 4.

5.2 Theorem 2.4

We begin with the proof of Theorem 2.4.

Proof of Theorem 2.4. First, we note that if the theorem were false, there would be a $\delta > 0$ and a ψ satisfying the hypotheses of Theorem 2.4 such that

$$||G'(u)\| \geq \psi(\|u\|) \tag{5.1}$$

when

$$u \in Q = \{u \in E : |G(u) - a| \leq 3\delta\}. \tag{5.2}$$

Reduce δ so that $3\delta < a - a_0$. Since $G \in C^1(E, \mathbb{R})$, for any $\theta \in (0, 1)$, there is a locally Lipschitz continuous mapping $Y(u)$ of $\hat{E} = \{u \in E : G'(u) \neq 0\}$ into E such that

$$\|Y(u)\| \leq 1, \quad \theta\|G'(u)\| \leq (G'(u), Y(u)), \quad u \in \hat{E} \tag{5.3}$$

(cf., e.g., [120]). Let

$$\begin{aligned}
Q_0 &= \{u \in E : |G(u) - a| \leq 2\delta\}, \\
Q_1 &= \{u \in E : |G(u) - a| \leq \delta\}, \\
Q_2 &= E \backslash Q_0, \\
\eta(u) &= d(u, Q_2)/[d(u, Q_1) + d(u, Q_2)].
\end{aligned}$$

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It is easily checked that $\eta(u)$ is locally Lipschitz continuous on E and satisfies

$$\begin{cases} \eta(u) = 1, & u \in Q_1, \\ \eta(u) = 0, & u \in \bar{Q}_2, \\ \eta(u) \in (0, 1), & \text{otherwise.} \end{cases} \tag{5.4}$$

Let $\rho(t) = 1/\psi(t)$. Then ρ is a positive, nondecreasing, locally Lipschitz continuous function on $[0, \infty)$ such that

$$\int_0^\infty \frac{d\tau}{\rho(\tau)} = \infty \tag{5.5}$$

by (2.8). Let

$$W(u) = -\eta(u)Y(u)\rho(\|u\|).$$

Then

$$\|W(u)\| \leq \rho(\|u\|), \quad u \in E.$$

By (5.5), for each $u \in E$, there is a unique solution of

$$\sigma'(t) = W(\sigma(t)), \quad t \in \mathbb{R}^+, \ \sigma(0) = u \tag{5.6}$$

(cf. Theorem 4.5). We have

$$\begin{aligned} dG(\sigma(t)u)/dt &= -\eta(\sigma(t)u)(G'(\sigma(t)u), Y(\sigma(t)u))\rho(\|\sigma(t)u\|) \\ &\leq -\theta\eta(\sigma)\|G'(\sigma)\|\rho(\|\sigma\|) \\ &\leq -\theta\eta(\sigma). \end{aligned} \tag{5.7}$$

Thus,

$$G(\sigma(t)u) \leq G(u), \quad t \geq 0,$$

$$G(\sigma(t)u) \leq a_0, \quad u \in A, \ t \geq 0,$$

and

$$\sigma(t)u = u, \quad u \in A, \ t \geq 0. \tag{5.8}$$

This follows from the fact that

$$G(\sigma(t)\,u) \leq a_0 < a - 3\delta, \quad u \in A, \ t \geq 0.$$

Hence, $\eta(\sigma(t)\,u) = 0$ for $u \in A, \ t \geq 0$. This means that

$$\sigma'(t)\,u = 0, \quad u \in A, \ t \geq 0,$$

and

$$\sigma(t)\,u = \sigma(0)\,u = u, \quad u \in A, \ t \geq 0.$$

Let

$$E_\alpha = \{u \in E : G(u) \leq \alpha\}. \tag{5.9}$$

There is a $T > 0$ such that

$$\sigma(T)E_{a+\delta} \subset E_{a-\delta}. \tag{5.10}$$

In fact, we can take $T > 2\delta/\theta$. To see this, let u be any element in $E_{a+\delta}$. If there is a $t_1 \in [0, T]$ such that $\sigma(t_1)u \notin Q_1$, then

$$G(\sigma(T)u) \leq G(\sigma(t_1)u) < a - \delta$$

by (5.7). Hence, $\sigma(T)u \in E_{a-\delta}$. On the other hand, if $\sigma(t)u \in Q_1$ for all $t \in [0, T]$, then $\eta(\sigma(t)u) = 1$ for all such t, and (5.7) yields

$$G(\sigma(T)u) \leq G(u) - \theta T < a - \delta. \tag{5.11}$$

Hence, (5.10) holds. Now, by (2.6), there is a $K \in \mathcal{K}$ such that

$$K \subset E_{a+\delta}. \tag{5.12}$$

As we saw, $\sigma(T) \in \Lambda(A)$. Let $\tilde{K} = \sigma(T)(K)$. Then $\tilde{K} \in \mathcal{K}$ by definition. But

$$\sup_{\tilde{K}} G = \sup_{u \in K} G(\sigma(T)u) < a - \delta,$$

which contradicts (2.6), proving the theorem. □

Next, we prove Theorem 2.8.

Proof. By (2.6) and (2.17), we see that $b_0 \leq a$. This implies (2.7) in view of (2.18). The result now follows from Theorem 2.4. □

Theorem 2.11 follows obviously from Theorem 2.8.

5.3 Theorem 2.12

Now we give the proof of Theorem 2.12.

Proof. If $a_0 < a$, the conclusion follows from Theorem 2.4. Assume that $a_0 = a$. If there did not exist a sequence satisfying (2.9), then there would be positive numbers δ, θ, T such that $2\delta < \theta T$ and (5.1) holds whenever $u \in Q$, where Q is given by (5.2). Since $a = a_0$, we see by (2.6), (2.17), and (2.20) that $b_0 = a$. Define Q_0, Q_1, Q_2, and $\eta(u)$ as before and let $\sigma(t)$ be the flow generated by the mapping (5.6). Let u be any element in $E_{a+\delta}$. If there is a $t_1 \leq T$ such that $\sigma(t_1)u \notin Q_1$, then

$$G(\sigma(t_1)u) < a - \delta. \tag{5.13}$$

On the other hand, if $\sigma(t)u \in Q_1$ for all $t \in [0, T]$, then $\eta(\sigma(t)u) \equiv 1$ in $[0, T]$ and

$$G(\sigma(t)u) \le G(u) - \theta t \le a + \delta - \theta t, \quad t \in [0, T]. \tag{5.14}$$

Thus we have

$$G(\sigma(T)u) < a - \delta. \tag{5.15}$$

Since $b_0 := \inf_B G = a$, this shows that

$$\sigma(T)E_{a+\delta} \cap B = \phi. \tag{5.16}$$

Moreover,

$$G(\sigma(t)u) \le a - \theta t, \quad u \in A,\ t \in [0, T]. \tag{5.17}$$

It therefore follows that (2.12), (2.13), and (2.21) hold for $b = a+\delta$. Let $K \in \mathcal{K}$ satisfy (5.12). Then, by hypothesis, there is a $\tilde{K} \in \mathcal{K}$ satisfying (2.14) with $b = a + \delta$. Now (2.21), (5.14), (5.16), and (5.17) imply that $\tilde{K} \cap B = \phi$, contradicting (2.17).

To prove the last statement, we take $\psi(t) = \rho(t) \equiv 1$. Still assuming $a_0 = a$, we note that if there did not exist a sequence satisfying both (1.4) and (2.22), then there would be positive numbers ϵ, δ, T such that $\delta < \epsilon T$ and (5.1) holds whenever

$$u \in Q = \{u \in E : d(u, B) \le 4T, |G(u) - a| \le 3\delta\}.$$

Let

$$\begin{aligned} Q_0 &= \{u \in E : d(u, B) \le 3T, |G(u) - a| \le 2\delta\}, \\ Q_1 &= \{u \in E : d(u, B) \le 2T, |G(u) - a| \le \delta\}. \end{aligned}$$

Since $a = b_0$, we see that $Q_1 \ne \phi$. Define Q_0, Q_1, Q_2, and $\eta(u)$ as before and let $\sigma(t)$ be the flow generated by the mapping

$$W(u) = -\eta(u)Y(u),$$

with everything now with respect to the new sets Q_j. In this case

$$\|W(u)\| \le 1. \tag{5.18}$$

Let u be any element in $E_{a+\delta}$. If there is a $t_1 \le T$ such that $\sigma(t_1)u \notin Q_1$, then either

$$G(\sigma(t_1)u) < a - \delta \tag{5.19}$$

or

$$d(\sigma(t_1)u, B) > 2T. \tag{5.20}$$

Since

$$\|\sigma(t)u - \sigma(t')u\| \le |t - t'|$$

by (5.18), (5.20) implies

$$d(\sigma(t)u, B) > T, \quad 0 \le t \le T. \tag{5.21}$$

On the other hand, if $\sigma(t)u \in Q_1$ for all $t \in [0, T]$, then

$$G(\sigma(T)u) \le G(u) - 2\epsilon T \le a + \delta - 2\delta = a - \delta. \tag{5.22}$$

Thus, either we have

$$G(\sigma(T)u) < a - \delta \tag{5.23}$$

or (5.21) holds. Since $b_0 = a$, this shows that

$$\sigma(T)E_{a+\delta} \cap B = \phi. \tag{5.24}$$

We also note that

$$\sigma(t)A \cap B = \phi, \quad 0 \le t \le T. \tag{5.25}$$

To see this, we observe, by (5.7), that

$$G(\sigma(t)u) \le a_0 - 2\epsilon \int_0^t \eta(\sigma(\tau)u)d\tau, \quad u \in A.$$

If $\sigma(t)u \in B$, we must have $G(\sigma(t)u) \ge b_0 \ge a_0$. The only way this can happen is if

$$\eta(\sigma(\tau)u) \equiv 0, \quad 0 \le \tau \le t.$$

But this implies $\sigma(\tau)u \in \bar{Q}_2$ for such τ, and this in turn implies either

$$G(\sigma(\tau)u) < a - \delta, \quad 0 \le \tau \le t,$$

or

$$d(\sigma(\tau)u, B) > 2T, \quad 0 \le \tau \le t.$$

In either case we cannot have $\sigma(t)u \in B$. Thus, (5.25) holds. Now, by (2.6), there is a $K \in \mathcal{K}$ such that

$$K \subset E_{a+\delta}, \tag{5.26}$$

and there is a $\tilde{K} \in \mathcal{K}$ such that (2.14) holds. By (5.25), $\tilde{K} \cap B = \phi$, contradicting the fact that A links B [mm]. This completes the proof of the theorem. □

Next, we give the proof of Theorem 2.6.

Proof. Define

$$B = \left\{ v \in \bigcup_{K \in \mathcal{K}} K \backslash A : G(v) \ge a_0 \right\}.$$

By (2.23), A links B relative to $\mathcal{K}$, and (2.20) holds. Apply Theorem 2.12. □

5.4 Theorem 2.14

Now we give the proof of Theorem 2.14.

Proof. We assume that $a = a_0$ and follow the proof of Theorem 2.4. In this case we cannot take $3\delta < a - a_0$ and we cannot conclude that (5.8) holds. Because of this, it does not follow that $\sigma(T)K \in \mathcal{K}$ for all $K \in \mathcal{K}$. Let $K \in \mathcal{K}$ be such that

$$a \leq \sup_{K \backslash A} G < a + \delta, \tag{5.27}$$

and let K_ρ be given by (2.25). Note that

$$a_0 \leq \sup_{K \backslash A} G, \quad K \in \mathcal{K}, \tag{5.28}$$

by (2.20). Since $G(u)$ is Lipschitz continuous on K_ρ and $a = a_0$, there is a constant C such that

$$G(u) \leq a + Cd(u), \quad u \in K_\rho, \tag{5.29}$$

where $d(u) = d(u, A)$. Pick $T > 2\delta/\theta$ and $T > \rho C/\theta$. As in the proof of Theorem 2.4, we have

$$\sigma(T)K \subset E_{a-\delta}. \tag{5.30}$$

Let $\zeta(t)$ be a nondecreasing, continuous function on $\mathbb{R}^+$ satisfying

$$\zeta(t) = \begin{cases} 0, & t = 0, \\ T, & t \geq \rho, \end{cases} \tag{5.31}$$

and

$$Tt/\rho < \zeta(t) < T, \quad 0 < t < \rho. \tag{5.32}$$

Note that the flow $\tilde{\sigma}(t) = \sigma(\zeta(t))$ satisfies (2.12) and (2.13). Consider the map

$$Su = \sigma(\zeta(d(u)))u, \quad u \in K. \tag{5.33}$$

Consequently, by hypothesis, $S(K) \in \mathcal{K}$. Since $d(u) = 0$ for $u \in A$, and $\sigma(0)u = u$, we have

$$Su = u, \quad u \in A.$$

If $u \in K \backslash K_\rho$, then $Su = \sigma(T)u \in E_{a-\delta}$. If $u \in K_\rho \backslash A$ and $Su \notin E_{a-\delta}$, then

$$\eta(\sigma(t)u) = 1, \quad 0 \leq t \leq \zeta(d(u)).$$

Consequently,

$$\begin{aligned} G(Su) &\leq G(u) - \theta\zeta(d(u)) \\ &\leq a + Cd(u) - \theta T d(u)/\rho \\ &< a. \end{aligned} \tag{5.34}$$

Thus,

$$G(Su) < a, \quad u \in K \backslash A,$$

contradicting (5.28). This completes the proof of the theorem. $\square$

Now, we prove Theorem 2.13.

Proof. Let B be given by (2.26). Then A links B relative to the system $\mathcal{K}$. Now apply Theorem 2.14. $\square$

5.5 Theorem 2.21

We can now prove Theorem 2.21.

Proof. We may assume that $a = a_0$. Otherwise, by Theorem 2.4, a sequence (2.9) exists with ψ replaced by $\tilde{\psi}(t) = \psi(t + \alpha)$. Since $\tilde{\psi}$ satisfies the hypotheses of Theorem 2.4 and

$$d(u, B') \leq \|u\| + \alpha,$$

for each $\delta > 0$, we can find a $u \in E$ such that

$$a - \delta \leq G(u) \leq a + \delta, \quad \|G'(u)\| < \tilde{\psi}(\|u\|) \leq \psi(d(u, B')),$$

which certainly implies (2.33).

If the conclusion of the theorem were not true, there would be a $\delta > 0$ such that

$$\psi(d(u, B')) \leq \|G'(u)\| \tag{5.35}$$

would hold for all u in the set

$$Q = \{u \in E : b_0 - 3\delta \leq G(u) \leq a + 3\delta\}. \tag{5.36}$$

By reducing δ if necessary, we can find $\theta < 1,\ T > 0$ such that

$$a_0 - b_0 + \delta < \theta T, \quad T \leq \int_{\delta+\alpha}^{R+\alpha} \psi(s)\, ds. \tag{5.37}$$

Thus, by Lemma 4.3, if $u(t)$ is the solution of (4.6) with $\rho(t) = 1/\psi(t),\ \gamma = 1$, $t_0 = 0$, and $u_0 = R$, then

$$u(t) \geq \delta, \quad t \in [0, T].$$

Let

$$Q_0 = \{u \in Q : b_0 - 2\delta \leq G(u) \leq a + 2\delta\}, \tag{5.38}$$

$$Q_1 = \{u \in Q : b_0 - \delta \leq G(u) \leq a + \delta\}, \tag{5.39}$$

and

$$Q_2 = E \backslash Q_0, \quad \eta(u) = d(u, Q_2)/[d(u, Q_1) + d(u, Q_2)]. \tag{5.40}$$

As before, we note that η satisfies (5.4). There is a locally Lipschitz continuous map $Y(u)$ of $\hat{E} = \{u \in E : G'(u) \neq 0\}$ into itself such that

$$\|Y(u)\| \leq 1, \quad \theta\|G'(u)\| \leq (G'(u), Y(u)), \quad u \in \hat{E} \tag{5.41}$$

(cf., e.g., [120]). Let $\sigma(t)$ be the flow generated by

$$W(u) = -\eta(u)Y(u)\rho(d(u, B')), \tag{5.42}$$

where $\rho(\tau) = 1/\psi(\tau)$. Since $\|W(u)\| \leq \rho(d(u, B')) \leq \tilde{\rho}(\|u\|) = 1/\tilde{\psi}(\|u\|)$ and is locally Lipschitz continuous, $\sigma(t)$ exists for all $t \in \mathbb{R}^+$ in view of Theorem 4.5. Since

$$\sigma(t)u - u = \int_0^t W(\sigma(\tau)u)d\tau, \tag{5.43}$$

we have

$$\|\sigma(t)u - \sigma(s)u\| \leq \int_s^t \rho(d(\sigma(r)u, B'))\, dr.$$

If $v \in B'$, we have

$$h(s) = d(\sigma(s)u, B') \leq \|\sigma(s)u - v\| \leq \|\sigma(t)u - v\| + \int_s^t \rho(d(\sigma(r)u, B'))\, dr,$$

which implies

$$h(s) \leq h(t) + \int_s^t \rho(h(r))\, dr. \tag{5.44}$$

We also have

$$\begin{aligned} dG(\sigma(t)u)/dt &= (G'(\sigma), \sigma') = -\eta(\sigma)(G'(\sigma), Y(\sigma))\rho(d(\sigma, B')) \\ &\leq -\theta\eta(\sigma)\|G'(\sigma)\|\rho(d(\sigma, B')) \\ &\leq -\theta\eta(\sigma)\psi(d(\sigma, B'))\rho(d(\sigma, B')) \\ &= -\theta\eta(\sigma) \end{aligned} \tag{5.45}$$

in view of (5.35) and (5.41). Now suppose $u \in E_{a+\delta}$ is such that there is a $t_1 \in [0, T]$ for which $\sigma(t_1)u \notin Q_1$. Then

$$G(\sigma(t_1)u) < b_0 - \delta,$$

since we cannot have $G(\sigma(t_1)u) > a + \delta$ for such u by (5.45). But this implies

$$G(\sigma(T)u) < b_0 - \delta. \tag{5.46}$$

On the other hand, if $\sigma(t)u \in Q_1$ for all $t \in [0, T]$, then

$$G(\sigma(T)u) \leq G(u) - \theta \int_0^T dt \leq a - \theta T < b_0 - \delta$$

by (5.37). Thus, (5.46) holds for $u \in E_{a+\delta}$ and, in particular, for any $u \in A$. By the definition (2.6) of a, there is a $K \in \mathcal{K}$ such that

$$\sup_K G < a + (\delta/2). \tag{5.47}$$

Note that $\sigma(t)u \neq v \in B$ for $u \in A$ and $t \in [0, T]$. For $v \in B'$, this follows from (5.44) and the fact that

$$h(t) = d(\sigma(t)u, B') \geq u(t) \geq \delta, \quad t \in [0, T],$$

in view of Lemma 4.8. If $v \in B \backslash B'$, we have, by (5.45),

$$G(\sigma(t)u) \leq a - \theta \int_0^t \eta(\sigma(\tau)u) d\tau < a, \ \ t > 0,$$

unless $\eta(\sigma(\tau)u) = 0$ for $0 \leq \tau \leq t$. But this would mean that $\eta(\sigma(\tau)u) \in \bar{Q}_2$ in view of (5.4). But then we would have $G(u) \leq b_0 - 2\delta$ since we cannot have $G(u) \geq a + 2\delta$. Thus,

$$G(\sigma(t)u) < a.$$

This shows that

$$B \cap \sigma(t)A = \phi, \quad 0 \leq t \leq T. \tag{5.48}$$

Taking $b = a + (\delta/2)$, (2.14) tells us that there is a $\tilde{K} \in \mathcal{K}$ such that

$$\tilde{K} \subset \bigcup_{t \in [0,T]} \sigma(t)A \ \cup \ \sigma(T)[E_b \cup K]. \tag{5.49}$$

But (5.46), (5.47), and (5.48) imply that $\tilde{K} \cap B = \phi$, contradicting the fact that A links B [mm]. $\square$

We also give the proof of Theorem 2.22.

Proof. Again, we may assume that $a = a_0$. We interchange A and B and consider the functional $\tilde{G}(u) = -G(u)$. Then

$$\tilde{a}_0 = \sup_B \tilde{G} = -\inf_B G = -b_0 < \infty$$

and

$$\tilde{b}_0 = \inf_A \tilde{G} = -\sup_A G = -a_0 > -\infty.$$

Moreover,

$$\tilde{a}_0 - \tilde{b}_0 = a_0 - b_0 < \int_\beta^{R+\beta} \psi(t)\, dt,$$

where

$$R \leq d'' = d(A'', B).$$

Since

$$A'' = \{u \in A : \tilde{G}(u) < \tilde{a}_0\},$$

we can apply Theorem 2.21 to conclude that for each $\delta > 0$, there is a $u \in E$ such that

$$\tilde{b}_0 - \delta \le \tilde{G}(u) \le \tilde{a}_0 + \delta, \quad \|\tilde{G}'(u)\| < \psi(d(u, A'')). \tag{5.50}$$

This implies (2.42). □

Theorems 2.24 and 2.19 follow from the same arguments used in the proof of Theorem 2.4 (in the case of Theorem 2.24 we substitute $-G$ for G). Now we give the proof of Theorem 2.18.

Proof. If the conclusion of the theorem were not true, then there would exist $\varepsilon > 0$ and $\psi(r) \in \Psi$ such that

$$\|G'(u)\| \ge \psi(\|u\|) \tag{5.51}$$

whenever

$$|G(u) - b| \le 3\varepsilon. \tag{5.52}$$

Let

$$Q = \{u \in E : |G(u) - b| \le 2\varepsilon\},$$
$$Q_1 = \{u \in E : |G(u) - b| \le \varepsilon\},$$
$$Q_2 = B \backslash Q,$$

and

$$\eta(u) = d(u, Q_2)/[d(u, Q_1) + d(u, Q_2)].$$

Note that $\eta(u)$ is locally Lipschitz continuous and satisfies

$$\begin{cases} \eta(u) = 1, & u \in Q_1, \\ \eta(u) = 0, & u \in \bar{Q}_2, \\ \eta(u) \in (0, 1), & \text{otherwise}. \end{cases} \tag{5.53}$$

Let $Y(u)$ be a locally Lipschitz continuous pseudo-gradient for G satisfying

$$G'(u)Y(u) \ge \alpha\|G'(u)\|, \quad \|Y(u)\| \le 1, \tag{5.54}$$

for some $\alpha > 0$. Let $W(u) = \eta(u)Y(u)$. Then $W(u)$ is a locally Lipschitz continuous mapping of E into itself such that

$$G'(u)W(u) \ge 0, \quad \|W(u)\| \le 1. \tag{5.55}$$

Let $\sigma(t)u$ be the solution of the initial-value problem

$$d\sigma(t)u/dt = W(\sigma(t)u), \quad \sigma(0)u = u. \tag{5.56}$$

The solution of (5.56) exists for every $u \in E$ and $t \geq 0$ by Theorem 4.5. By (5.55),

$$\|\sigma(t)u - u\| \leq t, \tag{5.57}$$

and by (5.54) and (5.56),

$$dG(\sigma(t)u)/dt = G'(\sigma(t)u)W(\sigma(t)u) \geq \alpha\eta(\sigma(t)u)\|G'(\sigma(t)u)\|. \tag{5.58}$$

This implies that

$$G(\sigma(t_1)u) \leq G(\sigma(t_2)u), \quad 0 \leq t_1 < t_2. \tag{5.59}$$

By hypothesis,

$$G(u) \geq b, \quad u \in \partial\omega_0. \tag{5.60}$$

Let

$$M = \sup_{u \in \partial\omega_0} \|u\| \tag{5.61}$$

and let T be given by

$$T = d(0, \partial\omega_0)/2. \tag{5.62}$$

Take $\varepsilon > 0$ so small that

$$2\varepsilon < \alpha \int_M^{T+M} \psi(t)\,dt. \tag{5.63}$$

By (5.57) and (5.62),

$$\sigma(t)u \neq 0, \quad u \in \partial\omega_0,\ 0 \leq t \leq T. \tag{5.64}$$

If $u \in \partial\omega_0$, but $u \notin Q_1$, we must have

$$G(u) > b + \varepsilon \tag{5.65}$$

since we cannot have

$$G(u) < b - \varepsilon \tag{5.66}$$

by (5.60). Thus, by (5.59),

$$G(\sigma(t)u) \geq G(u) > b + \varepsilon, \quad u \in \partial\omega_0,\ u \notin Q_1,\ 0 \leq t \leq T. \tag{5.67}$$

On the other hand, if $u \in \partial\omega_0 \cap Q_1$, let t_1 be the largest number not greater than T such that $\sigma(t)u \in Q_1$ for $0 \leq t \leq t_1$. If $t_1 < T$, then for $\delta > 0$ sufficiently small,

$$G(\sigma(t_1 + \delta)u) \geq G(\sigma(t_1)u) \geq b$$

and since $\sigma(t_1+\delta)u \notin Q_1$, we must have

$$G(\sigma(t_1+\delta)u) > b+\varepsilon.$$

Consequently,

$$G(\sigma(T)u) > b+\varepsilon. \tag{5.68}$$

If $t_1 = T$, then by (5.51), (5.53), (5.57), (5.58), (5.61),and (5.63),

$$\begin{aligned} G(\sigma(T)u) - G(u) &\geq \int_0^T \|G'(\sigma(t)u)\|\,dt \\ &\geq \alpha \int_0^T \psi(\|\sigma(t)u\|)\,dt \\ &\geq \alpha \int_0^T \psi(\|u\|+t)\,dt \\ &\geq \alpha \int_0^T \psi(M+t)\,dt \\ &= \alpha \int_M^{T+M} \psi(r)dr \\ &> 2\varepsilon. \end{aligned}$$

Thus, (5.68) holds as well in this case by (5.60). Consequently, (5.68) holds for all $u \in \partial\omega_0$. Let ω_T be the set of points $\sigma(T,u)$ where $u \in \omega_0$. Then ω_T is a bounded, open set in E with $\partial\omega_T$ consisting of those points of the form $\sigma(T)u,\ u \in \partial\omega_0$ by (5.57) and the continuous dependence of $\sigma(T)u$ on u. Since 0 is in Q_2 and $\eta \equiv 0$ there, we see that $\sigma(T)0 = 0$ by the uniqueness of solutions of (5.56). Thus, $0 = \sigma(T)0 \in \omega_T$. Thus, $\partial\omega_T \in \mathcal{K}$, and

$$G(u) > b+\varepsilon, \quad u \in \partial\omega_T, \tag{5.69}$$

by (5.68). But (5.69) contradicts (2.27). This completes the proof. □

In proving Theorem 2.20, we merely replace G by $-G$ and follow the proof of Theorem 2.18.

Chapter 6

Finding Linking Sets

6.1 Introduction

As we saw in Chapter 3, there are several sufficient conditions that imply that a set A links a set B in the sense of Definition 2.9. Our goal is to find all subsets that link according to this definition. At the present, we are very close to achieving this goal.

In the present chapter we define two relationships that are close to each other. The stronger one is sufficient for linking, while the weaker one is necessary. We use the following maps.

Definition 6.1. *We shall say that a map $\varphi : E \to E$ is of class Λ if it is a homeomorphism onto E and both φ, φ^{-1} are bounded on bounded sets. If, furthermore, $\varphi, \varphi^{-1} \in C^1(E; E)$, we shall say that $\varphi \in \Lambda_U$.*

Definition 6.2. *We shall say that a bounded set A is* **chained** *to a set B if $A \cap B = \phi$ and*

$$\inf_{x \in B} \|\varphi(x)\| \leq \sup_{x \in A} \|\varphi(x)\|, \quad \varphi \in \Lambda_U. \tag{6.1}$$

Definition 6.3. *We shall say that a bounded set A is* **strongly chained** *to a set B if $A \cap B = \phi$ and (6.1) holds for every $\varphi \in \Lambda$.*

Definition 6.4. *For $A \subset E$, we define*

$$\mathcal{K}_U(A) = \left\{ \varphi^{-1}(B_R) : \varphi \in \Lambda_U,\ R > \sup_{u \in A} \|\varphi(u)\| \right\},$$

where

$$B_R = \{u \in E : \|u\| < R\}.$$

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We shall show

Theorem 6.5. *If a bounded set A is strongly chained to a set B, then A links B.*

Theorem 6.6. *Let G be a $(C^2 \cap C_U)$-functional on E, and let A, B be nonempty subsets of E such that A is bounded and chained to B with*

$$a_0 := \sup_A G < b_0 := \inf_B G. \tag{6.2}$$

Let

$$a := \inf_{K \in \mathcal{K}_U(A)} \sup_K G. \tag{6.3}$$

Then there is a sequence $\{u_k\} \subset E$ such that

$$G(u_k) \to a, \;\; G'(u_k) \to 0. \tag{6.4}$$

Theorem 6.7. *If E is a Hilbert space and A links B, then it is chained to B.*

6.2 The strong case

Definition 6.8. *For $A \subset E$, we define*

$$\Lambda(A) = \{\varphi \in \Lambda : \varphi(u) = u, \;\; u \in A\}$$

and

$$\Lambda_U(A) = \{\varphi \in \Lambda_U : \varphi(u) = u, \;\; u \in A\}.$$

The following are easily proved.

Lemma 6.9.

$$\varphi_1, \varphi_2 \in \Lambda \Longrightarrow \varphi_1 \circ \varphi_2 \in \Lambda.$$

$$\varphi_1, \varphi_2 \in \Lambda_U \Longrightarrow \varphi_1 \circ \varphi_2 \in \Lambda_U.$$

$$\varphi_1, \varphi_2 \in \Lambda(A) \Longrightarrow \varphi_1 \circ \varphi_2 \in \Lambda(A).$$

$$\varphi_1, \varphi_2 \in \Lambda_U(A) \Longrightarrow \varphi_1 \circ \varphi_2 \in \Lambda_U(A).$$

Lemma 6.10.

$$\varphi \in \Lambda \;\text{ iff }\; \varphi^{-1} \in \Lambda.$$

$$\varphi \in \Lambda_U \;\text{ iff }\; \varphi^{-1} \in \Lambda_U.$$

$$\varphi \in \Lambda(A) \;\text{ iff }\; \varphi^{-1} \in \Lambda(A).$$

$$\varphi \in \Lambda_U(A) \;\text{ iff }\; \varphi^{-1} \in \Lambda_U(A).$$

Definition 6.11. *For $A \subset E$, we define*

$$\mathcal{K}(A) = \left\{ \varphi^{-1}(B_R) : \varphi \in \Lambda, \;\; R > \sup_{u \in A} \|\varphi(u)\| \right\}.$$

Lemma 6.12. *If $K \in \mathcal{K}(A)$ and $\sigma \in \Lambda(A)$, then $\sigma(K) \in \mathcal{K}(A)$.*

Proof. There is a $\varphi \in \Lambda$ such that $K = \varphi^{-1}(B_R)$ with $R > \sup_{u \in A} \|\varphi(u)\|$. Let

$$\tilde{\varphi} = \varphi \circ \sigma^{-1}.$$

Then $\tilde{\varphi} \in \Lambda$ (Lemma 6.9). Now,

$$\tilde{\varphi}^{-1} = \sigma \circ \varphi^{-1}.$$

Hence,

$$\sigma(K) = \sigma[\varphi^{-1}(B_R)] = \tilde{\varphi}^{-1}(B_R).$$

Moreover,

$$\tilde{\varphi}(u) = \varphi(u), \quad u \in A.$$

Hence,

$$\sup_{x \in A} \|\tilde{\varphi}(x)\| = \sup_{x \in A} \|\varphi(x)\| < R.$$

Therefore, $\sigma(K) \in \mathcal{K}(A)$. □

Corollary 6.13. *$\mathcal{K}(A)$ is a minimax system for A.*

Lemma 6.14. *If A is strongly chained to B and $K \in \mathcal{K}(A)$, then $K \cap B \neq \phi$.*

Proof. There is a $\varphi \in \Lambda$ such that $K = \varphi^{-1}(B_R)$ with $R > \sup_{u \in A} \|\varphi(u)\|$. Since A is strongly chained to B, we have

$$\inf_{x \in B} \|\varphi(x)\| \leq \sup_{x \in A} \|\varphi(x)\| < R.$$

Hence, there is a $v \in B$ such that $\varphi(v) \in B_R$. Thus, $v = \varphi^{-1}\varphi(v) \in \varphi^{-1}(B_R)$. □

Corollary 6.15. *If A is strongly chained to B, then A links B relative to $\mathcal{K}(A)$.*

Corollary 6.16. *If A is strongly chained to B, then A links B strongly.*

Proof. Theorem 2.11. □

We can now give the proof of Theorem 6.5.

Proof. Let G be a $(C^1 \cap C_U)$-functional on E, and let A, B be nonempty subsets of E such that A is strongly chained to B and (6.2) holds. Then

$$a := \inf_{K \in \mathcal{K}(A)} \sup_K G \tag{6.5}$$

is finite. By Lemma 6.14, $K \cap B \neq \phi$ for all $K \in \mathcal{K}(A)$. Hence, $a \geq b_0$. Since $a_0 < a$, then the theorem follows from Corollary 6.16. □

6.3 The remaining proofs

Lemma 6.17. *If A is chained to B and $K \in \mathcal{K}_U(A)$, then $K \cap B \neq \phi$.*

Proof. There is a $\varphi \in \Lambda_U$ such that $K = \varphi^{-1}(B_R)$ with $R > \sup_{u \in A} \|\varphi(u)\|$. Since A is chained to B, we have

$$\inf_{x \in B} \|\varphi(x)\| \leq \sup_{x \in A} \|\varphi(x)\| < R.$$

Hence, there is a $v \in B$ such that $\varphi(v) \in B_R$. Hence, $v = \varphi^{-1}\varphi(v) \in \varphi^{-1}(B_R)$. □

Lemma 6.18. *If $K \in \mathcal{K}_U(A)$ and $\sigma \in \Lambda_U(A)$, then $\sigma(K) \in \mathcal{K}_U(A)$.*

We can now give the proof of Theorem 6.6.

Proof. Let G be a $(C^2 \cap C_U)$-functional on E, and let A, B be nonempty subsets of E such that A is chained to B and (6.2) holds. Then a given by (6.3) is finite. By Lemma 6.17, $K \cap B \neq \phi$ for all $K \in \mathcal{K}_U(A)$. Hence, $a \geq b_0$. If (6.4) were false, there would exist a positive constant δ such that $3\delta < a - b_0$ and

(6.6) $$\|G'(u)\| \geq 3\delta$$

whenever

(6.7) $$u \in Q = \{u \in E : |G(u) - a| \leq 3\delta\}.$$

Let

$$\begin{aligned} Q_0 &= \{u \in E : |G(u) - a| \leq 2\delta\}, \\ Q_1 &= \{u \in E : |G(u) - a| \leq \delta\}, \\ Q_2 &= E \backslash Q_0, \\ \eta(u) &= d(u, Q_2)/[d(u, Q_1) + d(u, Q_2)]. \end{aligned}$$

It is easily checked that $\eta(u)$ is locally Lipschitz continuous on E and satisfies

$$\eta(u) = 1, \ u \in Q_1; \ \eta(u) = 0, \quad u \in \bar{Q}_2; \quad 0 < \eta(u) < 1, \ \text{ otherwise.}$$

Consider the differential equation

(6.8) $$\sigma'(t) = W(\sigma(t)), \quad t \in \mathbb{R}, \sigma(0) = u,$$

where

(6.9) $$W(u) = -\eta(u)G'(u)/\|G'(u)\|.$$

Since $G \in C^2(E, \mathbb{R})$, the mapping W is locally Lipschitz continuous on the whole of E and is bounded in norm by 1. Hence, by Corollary 4.6, (6.8) has a unique solution

for all $t \in \mathbb{R}$. Let us denote the solution of (6.8) by $\sigma(t)\,u$. The mapping $\sigma(t)$ is in $C^1_U(E \times \mathbb{R}, E)$ and is called the flow generated by (6.9). Note that

$$\begin{aligned} dG(\sigma(t)u)/dt &= (G'(\sigma(t)u), \sigma'(t)u) \\ &= -\eta(\sigma(t)u)\|G'(\sigma(t)u)\| \\ &\leq -2\delta\eta(\sigma(t)u). \end{aligned} \tag{6.10}$$

Thus,

$$G(\sigma(t)u) \leq G(u), \quad t \geq 0,$$
$$G(\sigma(t)u) \leq a_0, \quad u \in A,\ t \geq 0,$$

and

$$\sigma(t)u = u, \quad u \in A,\ t \geq 0.$$

Again, this follows from the fact that

$$G(\sigma(t)\,u) \leq a_0 \leq b_0 < a - 3\delta, \quad u \in A,\ t \geq 0.$$

Hence, $\eta(\sigma(t)\,u) = 0$ for $u \in A,\ t \geq 0$. This means that

$$\sigma'(t)\,u = 0, \quad u \in A,\ t \geq 0,$$

and

$$\sigma(t)\,u = \sigma(0)\,u = u, \quad u \in A,\ t \geq 0.$$

Let

$$E_\alpha = \{u \in E : G(u) \leq \alpha\}. \tag{6.11}$$

We note that there is a $T > 0$ such that

$$\sigma(T)E_{a+\delta} \subset E_{a-\delta}. \tag{6.12}$$

(In fact, we can take $T = 1$.) Let u be any element in $E_{a+\delta}$. If there is a $t_1 \in [0, T]$ such that $\sigma(t_1)u \notin Q_1$, then

$$G(\sigma(T)u) \leq G(\sigma(t_1)u) < a - \delta$$

by (6.10). Hence, $\sigma(T)u \in E_{a-\delta}$. On the other hand, if $\sigma(t)u \in Q_1$ for all $t \in [0, T]$, then $\eta(\sigma(t)u) = 1$ for all t, and (6.10) yields

$$G(\sigma(T)u) \leq G(u) - 2\delta T \leq a - \delta. \tag{6.13}$$

Hence, (6.12) holds. Now, by (6.3), there is a $K \in \mathcal{K}_U(A)$ such that

$$K \subset E_{a+\delta}. \tag{6.14}$$

Note that $\sigma(T) \in \Lambda_U(A)$. Let $\tilde{K} = \sigma(T)(K)$. Then $\tilde{K} \in \mathcal{K}_U(A)$ by Lemma 6.12. But

$$\sup_{\tilde{K}} G = \sup_{u \in K} G(\sigma(T)u) < a - \delta,$$

which contradicts (6.3), proving the theorem. $\square$

Now we give the proof of Theorem 6.7.

Proof. Assume that $\varphi \in \Lambda_U$ does not satisfy (6.1), and let $G(u) = \|\varphi(u)\|^2$. Then, by the definition of the class Λ_U, $G \in C^1_U(E, \mathbb{R})$, $\sup G(A) < \inf G(B)$, and G has no critical level $a \geq \inf_{x \in B} G(x) > 0$. To show the latter, assume that there is a sequence u_k satisfying

(6.15) $$G(u_k) \to a, \quad G'(u_k) \to 0.$$

Then we have, for any bounded sequence $v_k \in E$,

(6.16) $$(\varphi'(u_k)v_k, \varphi(u_k)) \to 0.$$

Let $v_k := (\varphi'(u_k))^{-1}(\varphi(u_k))$. Then $(\varphi'(u_k)v_k, \varphi(u_k)) = G(u_k) \to a$. However, the sequence v_k is bounded: $G(u_k) \to a$ implies that $\varphi(u_k)$ is bounded, which implies that u_k, and consequently that $\varphi'(u_k))^{-1}$ and v_k are also bounded. Hence, $G'(u_k) \to 0$ implies $G(u_k) \to 0$, showing that $a = 0$. Thus, A does not link B. □

6.4 Notes and remarks

The results of this chapter are from [138]. For related material cf. [137] and [156].

Chapter 7

Sandwich Pairs

7.1 Introduction

In this chapter we discuss the situation in which one cannot find linking sets that separate a functional G, i.e., satisfy

(7.1) $$a_0 := \sup_A G \leq b_0 := \inf_B G.$$

Are there weaker conditions that will imply the existence of a PS sequence

(7.2) $$G(u_k) \to a, \quad G'(u_k) \to 0?$$

Our answer is yes, and we find pairs of subsets such that the absence of (7.1) produces a PS sequence. We have

Definition 7.1. *We shall say that a pair of subsets A, B of a Banach space E forms a sandwich if, for any $G \in C^1(E, \mathbb{R})$, the inequality*

(7.3) $$-\infty < b_0 := \inf_B G \leq a_0 := \sup_A G < \infty$$

implies that there is a sequence satisfying

(7.4) $$G(u_k) \to c, \quad b_0 \leq c \leq a_0, \quad G'(u_k) \to 0.$$

Unlike linking, the order of a sandwich pair is immaterial; i.e., if the pair A, B forms a sandwich, so does B, A. Moreover, we allow sets forming a sandwich pair to intersect. One sandwich pair has been studied in Chapter 3. In fact, Theorem 3.17 tells us that if M, N are closed subspaces of a Hilbert space E, and $M = N^{\perp}$, then M, N form a sandwich pair if one of them is finite-dimensional.

Until recently, only complementing subspaces have been considered. The purpose of the present chapter is to show that other sets can qualify as well. Infinite-dimensional sandwich pairs will be considered in Chapter 15.

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7.2 Criteria

In this section we present sufficient conditions for sets to qualify as sandwich pairs. We have

Proposition 7.2. *If A, B is a sandwich pair and J is a diffeomorphism on the entire space having a derivative J' satisfying*

$$\|J'(u)^{-1}\| \leq C, \quad u \in E, \tag{7.5}$$

then JA, JB is a sandwich pair.

Proof. Suppose $G \in C^1$ satisfies

$$-\infty < b_0 := \inf_{JB} G \leq a_0 := \sup_{JA} G < \infty. \tag{7.6}$$

Let

$$G_1(u) = G(Ju), \quad u \in E.$$

Then

$$\begin{aligned} -\infty < b_0 := \inf_{JB} G = \inf_{Ju \in JB} G(Ju) = \inf_{B} G_1 \\ \leq a_0 := \sup_{JA} G = \sup_{Ju \in JA} G(Ju) = \sup_{A} G_1 < \infty. \end{aligned} \tag{7.7}$$

Since $A, \; B$ form a sandwich pair, there is a sequence $\{h_k\} \subset E$ such that

$$G_1(h_k) \to c, \quad b_0 \leq c \leq a_0, \quad G_1'(h_k) \to 0. \tag{7.8}$$

If we set $u_k = Jh_k$, this becomes

$$G(u_k) \to c, \quad b_0 \leq c \leq a_0, \quad G'(u_k)J'(h_k) \to 0. \tag{7.9}$$

In view of (7.5), this implies $G'(u_k) \to 0$. Thus, $JA, \; JB$ is a sandwich pair. □

Proposition 7.3. *Let N be a closed subspace of a Hilbert space E with complement $M' = M \oplus \{v_0\}$, where v_0 is an element in E having unit norm, and let δ be any positive number. Let $\varphi(t) \in C^1(\mathbb{R})$ be such that*

$$0 \leq \varphi(t) \leq 1, \quad \varphi(0) = 1,$$

and

$$\varphi(t) = 0, \quad |t| \geq 1.$$

Let

$$F(v + w + sv_0) = v + [s + \delta - \delta\varphi(\|w\|^2/\delta^2)]v_0, \quad v \in N, \; w \in M, \; s \in \mathbb{R}. \tag{7.10}$$

Assume that one of the subspaces M, N is finite-dimensional. Then $A = N' = N \oplus \{v_0\}$, $B = F^{-1}(\delta v_0)$ form a sandwich pair.

Proof. Define

$$J(v + w + sv_0) = v + w + [s + \delta - \delta\varphi(\|w\|^2/\delta^2)]v_0, \quad v \in N,\ w \in M,\ s \in \mathbb{R}.$$

Then J is a diffeomorphism on E with its inverse having a derivative satisfying (7.5). Moreover, $JA = N'$ and $JB = M + \delta v_0$. Hence, JA, JB form a sandwich pair as long as one of them is finite-dimensional (Theorem 3.17). We now apply Proposition 7.2. □

Theorem 7.4. *Let N be a finite dimensional subspace of a Banach space E. Let F be a Lipschitz continuous map of E onto N such that F = I on N and*

$$\|F(g) - F(h)\| \leq K\|g - h\|, \quad g, h \in E. \tag{7.11}$$

Let p be any point of N. Then A = N, B $= F^{-1}(p)$ *form a sandwich pair.*

Proof. Let G be a C^1-functional on E satisfying (7.3), where A, B are the subsets of E specified in the theorem. If the theorem is not true, then there is a $\delta > 0$ such that

$$\|G'(u)\| \geq 3\delta \tag{7.12}$$

whenever

$$b_0 - 3\delta \leq G(u) \leq a_0 + 3\delta. \tag{7.13}$$

Since $G \in C^1(E, \mathbb{R})$, there is a locally Lipschitz continuous mapping $Y(u)$ of $\hat{E} = \{u \in E : G'(u) \neq 0\}$ into E such that

$$\|Y(u)\| \leq 1, \quad u \in \hat{E},$$

and

$$(G'(u), Y(u)) \geq 2\delta$$

whenever u satisfies (7.13) (for the construction of such a map, cf., e.g., [112]). Let

$$\begin{aligned} Q_0 &= \{u \in E : b_0 - 2\delta \leq G(u) \leq a_0 + 2\delta\}, \\ Q_1 &= \{u \in E : b_0 - \delta \leq G(u) \leq a_0 + \delta\}, \\ Q_2 &= E \backslash Q_0, \\ \eta(u) &= \rho(u, Q_2)/[\rho(u, Q_1) + \rho(u, Q_2)]. \end{aligned}$$

It is easily checked that $\eta(u)$ is locally Lipschitz continuous on E and satisfies

$$\eta(u) = 1,\ u \in Q_1; \quad \eta(u) = 0,\ u \in \overline{Q}_2; \quad 0 < \eta(u) < 1,\ \text{otherwise}.$$

Consider the differential equation

$$\sigma'(t) = W(\sigma(t)), \quad t \in \mathbb{R},\ \sigma(0) = u \in N, \tag{7.14}$$

where

$$W(u) = -\eta(u)Y(u).$$

The mapping W is locally Lipschitz continuous on the whole of E and is bounded in norm by 1. Hence, by Theorem 4.5, (7.13) has a unique solution for all $t \in \mathbb{R}$. Let us denote the solution of (7.13) by $\sigma(t)u$. The mapping $\sigma(t)$ is in $C(E \times \mathbb{R}, E)$ and is called the flow generated by $W(u)$. Note that

$$\begin{aligned} dG(\sigma(t)u)/dt &= (G'(\sigma(t)u), \sigma'(t)u) \\ &= -\eta(\sigma(t)u)(G'(\sigma(t)u), Y(\sigma(t)u)) \\ &\leq -2\delta\eta(\sigma(t)u). \end{aligned} \tag{7.15}$$

Let

$$E_\alpha = \{u \in E : G(u) \leq \alpha\}.$$

I claim that there is a $T > 0$ such that

$$\sigma(T)E_{a_0+\delta} \subset E_{b_0-\delta}. \tag{7.16}$$

In fact, we can take $T > (a_0 - b_0 + \delta)/2\delta$. Let u be any element in $E_{a_0+\delta}$. If there is a $t_1 \in [0, T]$ such that $\sigma(t_1)u \notin Q_1$, then

$$G(\sigma(T)u) \leq G(\sigma(t_1)u) < b_0 - \delta$$

by (7.15). Hence, $\sigma(T)u \in E_{b_0-\delta}$. On the other hand, if $\sigma(t)u \in Q_1$ for all $t \in [0, T]$, then $\eta(\sigma(t)u) = 1$ for all t, and (7.15) yields

$$G(\sigma(T)u) \leq G(u) - 2\delta T \leq a_0 - 2\delta T \leq b_0 - \delta.$$

Hence, (7.16) holds. Let Ω be a bounded open subset of N containing the point p such that

$$\rho(\partial\Omega, p) > KT + \delta, \tag{7.17}$$

where ρ is the distance in E and K is the constant in (7.11). If $v \in \partial\Omega$, then

$$\|v - p\| \leq \|v - F\sigma(t)v\| + \|F\sigma(t)v - p\|.$$

Then

$$\|F\sigma(t)v - p\| > KT + \delta - tK > 0, \quad v \in \partial\Omega,\ 0 \leq t \leq T, \tag{7.18}$$

since

$$\|F\sigma(t)v - v\| \leq K \int_0^t \|\sigma'(s)v\|\, ds \leq Kt.$$

Let

$$H(t) = F\sigma(t).$$

Then $H(t)$ is a continuous map of $\overline{\Omega}$ into N for $0 \leq t \leq T$. Moreover, $H(t)v \neq p$ for $v \in \partial\Omega$ by (7.18). Hence, the Brouwer degree $d(H(t), \Omega, p)$ is defined. Consequently,

$$d(H(T), \Omega, p) = d(H(0), \Omega, p) = d(I, \Omega, p) = 1.$$

This means that there is a $v \in \overline{\Omega}$ such that

$$F\sigma(T)v = p.$$

But then

$$\sigma(T)v \in F^{-1}(p) = B.$$

This is not consistent with (7.16). Hence, A, B form a sandwich pair. $\square$

7.3 Notes and remarks

The material of this chapter comes from [135].

Chapter 8

Semilinear Problems

8.1 Introduction

In the present chapter we study several nonlinear boundary value problems that arise frequently in applications and illustrate the techniques described in the book.

8.2 Bounded domains

We assume that Ω is a bounded domain in $\mathbb{R}^n$ with boundary $\partial\Omega$ sufficiently regular so that the Sobolev inequalities hold and the embedding of $H^{m,2}(\Omega)$ in $L^2(\Omega)$ is compact (cf., e.g., [1]). Let A be a self-adjoint operator on $L^2(\Omega)$. We assume that $A \geq \lambda_0 > 0$ and that

$$C_0^\infty(\Omega) \subset D := D(A^{1/2}) \subset H^{m,2}(\Omega)$$

for some $m > 0$, where $C_0^\infty(\Omega)$ denotes the set of test functions in Ω (i.e., infinitely differentiable functions with compact supports in Ω) and $H^{m,2}(\Omega)$ denotes the Sobolev space. If m is an integer, the norm in $H^{m,2}(\Omega)$ is given by

$$\|u\|_{m,2} := \left(\sum_{|\mu| \leq m} \|D^\mu u\|^2 \right)^{1/2}. \tag{8.1}$$

Here D^μ represents the generic derivative of order $|\mu|$ and the norm on the right-hand side of (8.1) is that of $L^2(\Omega)$. We shall not assume that m is an integer. Let q be any number satisfying

$$\begin{aligned} 2 &\leq q \leq 2n/(n-2m), && 2m < n \\ 2 &\leq q < \infty, && n \leq 2m \end{aligned} \tag{8.2}$$

and let $f(x,t)$ be a Carathéodory function on $\Omega \times \mathbb{R}$. This means that $f(x,t)$ is continuous in t for a.e. $x \in \Omega$ and measurable in x for every $t \in \mathbb{R}$. We make the following assumption.

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(A) The function $f(x,t)$ satisfies

$$|f(x,t)| \le V(x)^q(|t|^{q-1} + W(x))$$

and

$$f(x,t)/V(x)^q = o(|t|^{q-1}) \text{ as } |t| \to \infty,$$

where $V(x) > 0$ is a function in $L^q(\Omega)$ such that

$$\|Vu\|_q \le C\|u\|_D, \quad u \in D, \tag{8.3}$$

and W is a function in $L^\infty(\Omega)$. Here

$$\|u\|_q := \left(\int_\Omega |u(x)|^q dx\right)^{1/q}$$

and

$$\|u\|_D := \|A^{1/2}u\|. \tag{8.4}$$

With the norm (8.4), D becomes a Hilbert space. Define

$$F(x,t) := \int_0^t f(x,s)\,ds$$

and

$$G(u) := \|u\|_D^2 - 2\int_\Omega F(x,u)\,dx. \tag{8.5}$$

It is readily shown that G is a continuously differentiable functional on the whole of D (cf., e.g., [122]). Since the embedding of D in $L^2(\Omega)$ is compact, the spectrum of A consists of isolated eigenvalues of finite multiplicity:

$$0 < \lambda_0 < \lambda_1 < \cdots < \lambda_\ell < \cdots.$$

Let $\lambda_\ell,\ \ell > 0$, be one of these eigenvalues. We assume that the eigenfunctions of λ_ℓ are in $L^\infty(\Omega)$ and that the following hold:

$$2F(x,t) \le \lambda_\ell t^2 + W_1(x), \quad x \in \Omega,\ t \in \mathbb{R} \tag{8.6}$$

for some $W_1(x) \in L^1(\mathbb{R})$,

$$\lambda_\ell t^2 \le 2F(x,t), \quad |t| \le \delta, \tag{8.7}$$

for some $\delta > 0$,

$$\nu t^2 \le 2F(x,t), \quad x \in \Omega,\ t \in \mathbb{R}, \tag{8.8}$$

for some $\nu > \lambda_{\ell-1}$,

$$H(x,t) := 2F(x,t) - tf(x,t) \le C(|t| + 1), \tag{8.9}$$

and

$$\sigma(x) := \limsup_{|t|\to\infty} H(x,t)/|t| < 0 \quad \text{a.e.} \tag{8.10}$$

We wish to obtain a solution of

$$Au = f(x,u), \quad u \in D. \tag{8.11}$$

By a solution of (8.11) we shall mean a function $u \in D$ such that

$$(u,v)_D = (f(\cdot,u),v), \quad v \in D. \tag{8.12}$$

If $f(x,u)$ is in $L^2(\Omega)$, then a solution of (8.12) is in $D(A)$ and solves (8.11) in the classical sense. Otherwise we call it a weak (or semi-strong) solution. We have

Theorem 8.1. *Under the above hypotheses,*

$$Au = f(x,u), \quad u \in D \tag{8.13}$$

has at least one nontrivial solution.

Proof. Let N denote the subspace of $L^2(\Omega)$ spanned by the eigenfunctions of A corresponding to the eigenvalues $\lambda_0,\ldots,\lambda_\ell$, and let $M = N^\perp \cap D$. Thus, $D = M \oplus N$. This time we take

$$G(u) = 2\int F(x,u)\,dx - \|u\|_D^2,$$

the negative of (8.5). We are therefore looking for solutions of $G'(u) = 0$. Let N' be the set of those functions in N that are orthogonal to $E(\lambda_\ell)$. It is spanned by those eigenfunctions corresponding to $\lambda_0,\ldots,\lambda_{\ell-1}$. Let v_0 be an eigenfunction of λ_ℓ with norm 1. Let $M_1 = M \oplus E(\lambda_\ell) \ominus \{v_0\}$. We can write

$$E = M_1 \oplus \{v_0\} \oplus N'.$$

Consider the mapping

$$F(v + w + sv_0) = w + [s + \rho - \rho\varphi(\|v\|^2/\rho^2)]v_0, \quad v \in N,\ w \in M_1,\ s \in \mathbb{R},$$

where φ satisfies the hypotheses of Proposition 7.3 and $\rho > 0$ is to be chosen. We take

$$A = M_1 \oplus \{v_0\}, \quad B = F^{-1}(\rho v_0).$$

By Proposition 7.3, A, B form a sandwich pair.

For $v \in N$, we write $v = v' + y$, where $v' \in N'$ and $y \in E(\lambda_\ell)$. Since $E(\lambda_\ell)$ is finite-dimensional and contained in $L^\infty(\Omega)$, there is a $\rho > 0$ such that

$$\|y\|_D \le \rho \text{ implies } \|y\|_\infty \le \delta/2, \tag{8.14}$$

where δ is given by (8.7). Thus, if

$$\|v\|_D \le \rho \quad \text{and} \quad |v(x)| \ge \delta, \tag{8.15}$$

then

$$\delta \le |v(x)| \le |v'(x)| + |y(x)| \le |v'(x)| + \delta/2.$$

Hence,

$$|v(x)| \le 2|v'(x)|$$

holds for all $x \in \bar{\Omega}$ satisfying (8.15). Thus by (8.7)

$$\begin{aligned} G(v) &\ge \lambda_\ell \int_{|v|<\delta} v^2 dx - 2\int_{|v|>\delta} \{|Vv|^q + |V^q v|W\} dx - \|v\|_D^2 \\ &\ge \lambda_\ell \|v\|^2 - \lambda_\ell \int_{|v|>\delta} v^2 dx - \|v\|_D^2 - C\int_{2|v'|>\delta} \{|Vv'|^q + \delta^{1-q}|Vv'|^q\} dx \\ &\ge \lambda_\ell \|v\|^2 - \|v\|_D^2 - C\int_{2|v'|>\delta} \{|Vv'|^q + \delta^{1-q}|Vv'|^q + \delta^{2-q}|v'|^q\} dx \\ &\ge \lambda_\ell \|v'\|^2 - \|v'\|_D^2 - C\|v'\|_D^q \\ &\ge \left(\frac{\lambda_\ell}{\lambda_{\ell-1}} - 1 - C\|v'\|_D^{q-2}\right)\|v'\|_D^2. \end{aligned}$$

To see this, note that when (8.15) holds, we have $|Vv| \le 2|Vv'|$ and

$$|V^q v| \le V^q |v| \frac{|v|^{q-1}}{\delta^{q-1}} \le \delta^{1-q} V^q |2v'|^q.$$

Moreover,

$$\|v'\|_q \le C'\|v'\|_{m,2} \le C''\|v'\|_D$$

by the Sobolev inequality and the embedding of D in $H^{m,2}(\Omega)$. From this we see that there are positive constants ϵ, ρ such that

$$G(v) \ge \epsilon\|v'\|_D^2, \quad \|v\|_D \le \rho,\ v \in N_\ell.$$

Moreover, this shows that for each positive $\rho_1 \le \rho$,

$$G(v) \ge \epsilon_1, \quad \|v\|_D = \rho_1,\ v \in N_\ell, \tag{8.16}$$

for some positive ϵ_1 unless there is a solution of

$$Ay = \lambda_\ell y = f(x, y), \quad y \in E(\lambda_\ell)\backslash\{0\}$$

(cf. [122]). Since such a solution would solve (8.13), we may assume that (8.16) holds.

Since

$$\|v\|_D^2 \le \lambda_\ell \|v\|^2, \quad v \in N,$$

and

$$\lambda_{\ell+1}\|w\|^2 \le \|w\|_D^2, \quad w \in M,$$

we have, by (8.6),

$$G(w) \le \lambda_\ell \|w\|^2 + B_1 - \|w\|_D^2 \le B_1, \quad w \in A,$$

where $B_1 = \int_\Omega W_1(x)\,dx$. Moreover, (8.8) implies

$$G(v') \geq (\nu - \lambda_{\ell-1})\|v'\|^2, \quad v' \in N'.$$

Hence, there is an $\varepsilon > 0$ such that

$$G(v) \geq \varepsilon, \quad v \in B.$$

In view of these inequalities, we can now apply Proposition 7.3 to conclude that there is a sequence $\{u_k\} \subset D$ such that

$$G(u_k) \to c, \quad \varepsilon \leq c \leq B_1, \quad G'(u_k) \to 0. \tag{8.17}$$

Let $\rho_k = \|u_k\|_D$. If $\rho_k \to \infty$, then

$$G(u_k) = 2\int_\Omega F(x, u_k)\,dx - \rho_k^2 \to c \tag{8.18}$$

and

$$(G'(u_k), u_k)/2 = \int_\Omega f(x, u_k)u_k\,dx - \rho_k^2 = o(\rho_k).$$

Hence,

$$\int_\Omega H(x, u_k)\,dx = o(\rho_k).$$

Let $\tilde{u_k} = u_k/\rho_k$. Then $\|\tilde{u}_k\|_D = 1$. Thus, there is a renamed subsequence such that $\tilde{u}_k \to \tilde{u}$ weakly in D, strongly in $L^2(\Omega)$, and a.e. in Ω. By (8.9) and (8.10),

$$\begin{aligned}\limsup \int_\Omega H(x, u_k)\,dx/\rho_k &\leq \int_\Omega \limsup[H(x, u_k)/|u_k|]|\tilde{u}_k|\,dx \\ &= \int_\Omega \sigma(x)|\tilde{u}|\,dx.\end{aligned}$$

Since $\sigma(x) < 0$ a.e. in Ω, the last two statements imply that $\tilde{u} \equiv 0$. However, we see from (8.18) that

$$2\int_\Omega F(x, u_k)\,dx/\rho_k^2 \to 1,$$

while (8.6) implies

$$\limsup 2\int_\Omega F(x, u_k)\,dx/\rho_k^2 \leq \lambda_\ell \int_\Omega \tilde{u}^2\,dx,$$

showing that $\tilde{u} \not\equiv 0$. This contradiction tells us that the ρ_k must be bounded. We can now apply Theorem 3.4.1 of [122] to conclude that there is a $u \in D$ satisfying

$$G(u) = c, \quad G'(u) = 0. \tag{8.19}$$

Since $c \geq \varepsilon > 0$, we see that $u \neq 0$, and the proof is complete. $\square$

The proof of Theorem 8.1 implies

Corollary 8.2. *If λ_ℓ is a simple eigenvalue, then hypothesis (8.6) in Theorem 8.1 can be weakened to*

$$2F(x,t) \leq \lambda_{\ell+1}t^2 + W_1(x), \quad x \in \Omega,\ t \in \mathbb{R}, \tag{8.20}$$

for some $W_1(x) \in L^1(\mathbb{R})$.

Remark 8.3. *The proof of Theorem 8.1 is much simpler if $\ell = 0$. In this case $N' = \{0\}$ and (8.14) immediately implies (8.16). The rest of the proof is unchanged.*

We now show that we can essentially reverse the inequalities (8.6)–(8.10) and obtain the same results. In fact, we have

Theorem 8.4. *Equation (8.13) has at least one nontrivial solution if we assume $\ell > 0$ and*

$$\lambda_\ell t^2 \leq 2F(x,t) + W_1(x), \quad x \in \Omega,\ t \in \mathbb{R}, \tag{8.21}$$

for some $W_1(x) \in L^1(\mathbb{R})$,

$$2F(x,t) \leq \lambda_\ell t^2, \quad |t| \leq \delta, \tag{8.22}$$

for some $\delta > 0$,

$$2F(x,t) \leq \nu t^2, \quad x \in \Omega,\ t \in \mathbb{R}, \tag{8.23}$$

for some $\nu < \lambda_{\ell+1}$,

$$H(x,t) \geq -C(|t|+1), \quad x \in \Omega,\ t \in \mathbb{R}, \tag{8.24}$$

and

$$\liminf_{|t|\to\infty} H(x,t)/|t| > 0 \quad \text{a.e.} \tag{8.25}$$

Proof. In this case we take G to be the functional (8.5). We take $A = N$, $N' = N \ominus \{v_0\}$, and consider the mapping

$$F(v + w + sv_0) = v + [s + \delta - \delta\varphi(\|w\|^2/\delta^2)]v_0, \quad v \in N',\ w \in M,\ s \in \mathbb{R},$$

where φ satisfies the hypotheses of Proposition 7.3. By (8.21), we have

$$G(v) \leq B_1, \quad v \in N.$$

For $w \in M_1$, we write $w = w' + y$, where $w' \in M$ and $y \in E(\lambda_\ell)$. Then (8.22) implies

$$G(w) \geq \varepsilon_1, \quad \|w\|_D = \rho,\ w \in M_1,$$

unless (8.13) has a nontrivial solution. Hence, by the argument given in the proof of Theorem 8.1 we have a sequence satisfying (8.17). If $\tilde{u}_k$ and $\tilde{u}$ are as in the proof of

Theorem 8.1, then (8.24), (8.25) imply that $\tilde{u} \equiv 0$ as in that proof. However, (8.18) implies

$$2\int_\Omega F(x, u_k)\,dx/\rho_k^2 \to 1,$$

while (8.23) implies

$$\limsup 2\int_\Omega F(x, u_k)\,dx/\rho_k^2 \le \nu \int_\Omega \tilde{u}^2 dx,$$

showing that $\tilde{u} \not\equiv 0$. This contradiction proves the theorem as in the case of Theorem 8.1. □

The proof of Theorem 8.4 implies

Corollary 8.5. *If λ_ℓ is a simple eigenvalue, then hypothesis (8.21) in Theorem 8.4 can be weakened to*

$$\lambda_{\ell-1}t^2 \le 2F(x,t) + W_1(x), \quad x \in \Omega,\ t \in \mathbb{R},\ \text{for some } W_1(x) \in L^1(\mathbb{R}). \tag{8.26}$$

8.3 Some useful quantities

We now show how we can improve the results of the last section. For each fixed k, let N_k denote the subspace of $D := D(A^{1/2})$ spanned by the eigenfunctions corresponding to $\lambda_0, \dots, \lambda_k$, and let $M_k = N_k^\perp \cap D$. Then $D = M_k \oplus N_k$. We define

$$\alpha_k := \max\{(Av, v) : v \in N_k, v \ge 0, \|v\| = 1\}, \tag{8.27}$$

where $\|v\|$ denotes the $L^2(\Omega)$-norm of v. We assume that A has an eigenfunction φ_0 of constant sign a.e. on Ω corresponding to the eigenvalue λ_0.

Next we define for $a \in \mathbb{R}$

$$\gamma_k(a) := \max\{(Av, v) - a\|v^-\|^2 \ :\ v \in N_k,\ \|v^+\| = 1\} \tag{8.28}$$

and

$$\Gamma_k(a) := \inf\{(Aw, w) - a\|w^-\|^2 :\ w \in M_k,\ \|w^+\| = 1\}, \tag{8.29}$$

where $u^\pm = \max\{\pm u, 0\}$.

We take any integer $\ell \ge 0$ and let N denote the subspace of $L^2(\Omega)$ spanned by the eigenspaces of A corresponding to the eigenvalues $\lambda_0, \lambda_1, \dots, \lambda_\ell$. We take $M = N^\perp \cap D$, where $D = D(A^{1/2})$. We assume that $F(x,t)$ satisfies

$$\begin{aligned} a_1(t^-)^2 + \gamma_\ell(a_1)(t^+)^2 - W_1(x) &\le 2F(x,t) \\ &\le a_2(t^-)^2 + \nu(t^+)^2, \quad x \in \Omega,\ t \in \mathbb{R}, \end{aligned} \tag{8.30}$$

for numbers a_1, a_2 satisfying $\alpha_\ell < a_1 \le a_2$, where W_1 is a function in $L^1(\Omega)$ and $\nu < \Gamma_\ell(a_2)$. We also assume that

$$2F(x,t) \le \lambda_{\ell+1}t^2, \quad |t| \le \delta \tag{8.31}$$

for some $\delta > 0$,

$$|f(x,t)| \le C|t| + W(x), \quad W \in L^2(\Omega), \tag{8.32}$$

$$f(x,t)/t \to \alpha_\pm(x) \text{ a.e. as } t \to \pm\infty, \tag{8.33}$$

and the only solution of

$$Au = \alpha_+(x)u^+ - \alpha_-(x)u^- \tag{8.34}$$

is $u \equiv 0$. We have

Theorem 8.6. *Under the above hypotheses, (8.13) has a nontrivial solution.*

Proof. By (8.28),

$$\|v\|_D^2 \le a_1\|v^-\|^2 + \gamma_\ell(a_1)\|v^+\|^2, \quad v \in N, \tag{8.35}$$

and by (8.29), we have

$$a_2\|w^-\|^2 + \Gamma_\ell(a_2)\|w^+\|^2 \le \|w\|_D^2, \quad w \in M. \tag{8.36}$$

Hence,

$$G(v) \le B_1, \quad v \in N. \tag{8.37}$$

Since $\nu < \Gamma_\ell(a_2)$, we see by continuity that there is an $\varepsilon > 0$ such that

$$\nu < (1-\varepsilon)\Gamma_\ell\left(\frac{a_2}{1-\varepsilon}\right).$$

Hence,

$$\begin{aligned} G(w) &\ge \varepsilon\|w\|_D^2 + (1-\varepsilon)\left[\Gamma_\ell\left(\frac{a_2}{1-\varepsilon}\right) - \frac{\nu}{1-\varepsilon}\right]\|w^+\|^2 \\ &\ge \varepsilon\|w\|_D^2, \quad w \in M, \end{aligned} \tag{8.38}$$

by (8.30).

As in the proof of Theorem 8.1, we note that the following alternative holds:

Either

(a) there is an infinite number of eigenfunctions $y \in E(\lambda_\ell)\backslash\{0\}$ such that

$$Ay = f(x,y) = \lambda_\ell y, \tag{8.39}$$

or

(b) for each $\rho > 0$ sufficiently small, there is an $\varepsilon > 0$ such that

$$G(w) \ge \varepsilon, \quad \|w\|_D = \rho,\ w \in M_\ell. \tag{8.40}$$

Since option (a) solves our problem, we may assume that option (b) holds. Let $v_0 \in E(\lambda_\ell)$, and let F be the mapping (7.10). Take $A = N$, $B = F^{-1}(\delta v_0)$. By (8.37), (8.38), and (8.40) we see that (7.3) holds with $b_0 > 0$ and $a_0 = B_1$. By Proposition 7.3, we can conclude that there is a sequence $\{u_k\} \subset D$ such that

$$G(u_k) \to c, \quad b_0 \leq c \leq B_1, \quad G'(u_k) \to 0. \tag{8.41}$$

Thus,

$$G(u_k) = \|u_k\|_D^2 - 2\int_\Omega F(x, u_k)dx \to c \tag{8.42}$$

and

$$(G'(u_k), v) = 2(u_k, v)_D - 2(f(u_k), v) \to 0, \quad v \in D. \tag{8.43}$$

If $\rho_k = \|u_k\|_D \to \infty$, let $\tilde{u}_k = u_k/\rho_k$. Then $\|\tilde{u}_k\|_D = 1$. Thus, there is a renamed subsequence such that $\tilde{u}_k \to \tilde{u}$ weakly in D, strongly in $L^2(\Omega)$, and a.e. in Ω. Hence,

$$G(u_k)/\rho_k^2 = \|\tilde{u}_k\|_D^2 - 2\int_\Omega F(x, u_k)dx/\rho_k^2 \to 0.$$

Since

$$|F(x, u_k)|/\rho_k^2 \leq C(|\tilde{u}_k(x)|^2 + W_3(x)/\rho_k^2), \quad W_3 \in L^1(\Omega), \tag{8.44}$$

by (8.30), and the right-hand side of (8.44) converges to $C|\tilde{u}(x)|^2$ in $L^1(\Omega)$ and

$$2F(x, u_k(x))/\rho_k^2 \to \alpha_+(x)(\tilde{u}^+)^2 + \alpha_-(x)(\tilde{u}^-)^2 \quad \text{a.e.}, \tag{8.45}$$

we see that the convergence in (8.45) is not only pointwise a.e., but also in $L^1(\Omega)$. Since $\|\tilde{u}_k\|_D = 1$, (8.44) implies

$$\int_\Omega \{\alpha_+(\tilde{u}^+)^2 + \alpha_-(\tilde{u}^-)^2\}dx = 1. \tag{8.46}$$

Also,

$$(G'(u_k), v)/\rho_k = 2(\tilde{u}_k, v)_D - 2(f(u_k), v)/\rho_k \to 0$$

for each $v \in D$. This implies

$$(\tilde{u}, \tilde{v})_D = (\alpha_+\tilde{u}^+ - \alpha_-\tilde{u}^-, v), \quad v \in D.$$

Consequently, $\tilde{u}$ is a solution of (8.34). By hypothesis, $\tilde{u} \equiv 0$. But this contradicts (8.46). Hence, $\rho_k \leq C$. The theorem now follows from Theorem 3.4.1 of [122]. $\square$

8.4 Unbounded domains

Now we allow the domain $\Omega \subset \mathbb{R}^n$ to be unbounded. Assume hypothesis (A), and assume that

$$H(x, t) = 2F(x, t) - tf(x, t) \geq -W_3(x) \in L^1(\Omega), \quad x \in \Omega,\ t \in \mathbb{R}, \tag{8.47}$$

and

$$H(x,t) \to \infty \text{ a.e. as } |t| \to \infty. \tag{8.48}$$

We have

Theorem 8.7. *Assume that the spectrum of A consists of isolated eigenvalues of finite multiplicity*

$$0 < \lambda_0 < \lambda_1 < \cdots < \lambda_k < \cdots, \tag{8.49}$$

and let ℓ be a nonnegative integer. Take N to be the subspace of D spanned by the eigenspaces of A corresponding to the eigenvalues $\lambda_0, \lambda_1, \ldots, \lambda_\ell$. We take $M = N^\perp \cap D$. Assume that there are functions $W_1,\ W_2 \in L^1(\Omega)$ and numbers a_1, a_2 such that $\alpha_\ell < a_1 \le a_2$ and

$$\begin{aligned} a_1(t^-)^2 + \gamma_\ell(a_1)(t^+)^2 - W_1(x) &\le 2F(x,t) \\ &\le a_2(t^-)^2 + \Gamma_\ell(a_2)(t^+)^2 + W_2(x), \quad x \in \Omega,\ t \in \mathbb{R}, \end{aligned} \tag{8.50}$$

and that (8.47) and (8.48) hold. Then (8.13) has at least one solution.

Proof. First, we note that

$$\sup_N G \le B_1, \quad \inf_M G \ge -B_2, \quad B_j = \int_\Omega W_j(x)dx. \tag{8.51}$$

To see this, note that by (8.28) we have

$$\|v\|_D^2 \le a_1\|v^-\|^2 + \gamma_\ell(a_1)\|v^+\|^2, \quad v \in N. \tag{8.52}$$

By (8.29) we have

$$a_2\|w^-\|^2 + \Gamma_\ell(a_2)\|w^+\|^2 \le \|w\|_D^2, \quad w \in M. \tag{8.53}$$

Hence,

$$G(v) \le B_1, \quad v \in N,$$

and

$$G(w) \ge -B_2, \quad w \in M,$$

by (8.50). By Theorem 3.19, we conclude that for any sequence $R_k \to \infty$, there is a sequence $\{u_k\} \subset D$ such that

$$G(u_k) \to c, \quad -B_2 \le c \le B_1, \quad (R_k + \|u_k\|_D)\|G'(u_k)\| \le \frac{B_1 + B_2}{\ln(4/3)}. \tag{8.54}$$

In particular, we have

$$\|u_k\|_D^2 - 2\int_\Omega F(x,u_k)\,dx \to c \tag{8.55}$$

and

$$\left| \|u_k\|_D^2 - (f(\cdot, x_k), u_k) \right| \le K. \tag{8.56}$$

Consequently,

$$\left| \int_\Omega H(x, u_k)\, dx \right| \le K'. \tag{8.57}$$

If $\rho_k = \|u_k\|_D \to \infty$, let $\tilde{u}_k = u_k/\rho_k$. Then $\|\tilde{u}_k\|_D = 1$. Consequently, there is a renamed subsequence such that $\tilde{u}_k \to \tilde{u}$ weakly in D, strongly in $L^2(\Omega)$, and a.e. in Ω. In view of (8.55), we have

$$1 - 2 \int_\Omega F(x, u_k)/\rho_k^2\, dx \to 0. \tag{8.58}$$

But by (8.50), we have

$$2F(x, u_k)/\rho_k^2 \le a_2(\tilde{u}_k^-)^2 + \Gamma_\ell(a_2)(\tilde{u}_k^+)^2 + W_2(x)/\rho_k^2. \tag{8.59}$$

In the limit this implies

$$1 \le a_2 \|\tilde{u}^-\|^2 + \Gamma_\ell(a_2)\|\tilde{u}^+\|^2.$$

This shows that $\tilde{u} \not\equiv 0$. Let Ω_0 be the subset of Ω on which $\tilde{u} \neq 0$. Then

$$|u_k(x)| = \rho_k |\tilde{u}_k(x)| \to \infty, \quad x \in \Omega_0. \tag{8.60}$$

If $\Omega_1 = \Omega \backslash \Omega_0$, then we have

$$\int_\Omega H(x, u_k)\, dx = \int_{\Omega_0} + \int_{\Omega_1} \ge \int_{\Omega_0} H(x, u_k)\, dx - \int_{\Omega_1} W_1(x)\, dx \to \infty. \tag{8.61}$$

This contradicts (8.57), and we see that $\rho_k = \|u_k\|_D$ is bounded. Once we know that the ρ_k are bounded, we can apply Theorem 3.4.1 of [122] to obtain the desired conclusion. □

Remark 8.8. *It should be noted that the crucial element in the proof of Theorem 8.7 was (8.56). If we had been dealing with an ordinary Palais–Smale sequence, we could only conclude that*

$$\|u_k\|_D^2 - (f(\cdot, u_k), u_k) = o(\rho_k),$$

which would imply only that

$$\int_\Omega H(x, u_k) dx = o(\rho_k).$$

This would not contradict (8.61), and the argument would not go through.

We also have

Theorem 8.9. *The conclusion of Theorem 8.7 holds if in place of (8.47), (8.48) we assume that*

$$H(x,t) \le W_1(x) \in L^1(\Omega), \quad x \in \Omega,\ t \in \mathbb{R}, \tag{8.62}$$

and

$$H(x,t) \to -\infty \text{ a.e. as } |t| \to \infty. \tag{8.63}$$

Proof. We use (8.62) and (8.63) to replace (8.61) with

$$\int_\Omega H(x,u_k)\,dx = \int_{\Omega_0} + \int_{\Omega_1} \le \int_{\Omega_0} H(x,u_k)\,dx + \int_{\Omega_1} W_1(x)\,dx \to -\infty. \tag{8.64}$$

We then proceed as before. □

As a specific example of an operator A satisfying the hypotheses of Theorems 8.7 and 8.9, let $g(x)$ be a measurable function satisfying

$$g(x) \ge c_0 > 0, \quad x \in \mathbb{R}^n,$$

for some positive constant c_0. We consider the problem

$$-\Delta u(x) + g(x)^2 u(x) = f(x,u(x)), \quad x \in \mathbb{R}^n. \tag{8.65}$$

We define the operator A on $L^2 = L^2(\mathbb{R}^n)$ by $u \in D(A)$ and $Au = f$ if $u \in D = H^{1,2} = H^{1,2}(\mathbb{R}^n)$ and

$$(u,v)_D = (\nabla u, \nabla v) + (gu, gv) = (f,v), \quad v \in H^{1,2}.$$

We assume that $g(x) \in L^2_{\text{loc}}$ and that multiplication by g^{-1} is a compact operator from $H^{1,2}$ to L^2. It follows that A is a self-adjoint operator on L^2 that is bijective. Moreover, A^{-1} is a compact operator on L^2. It follows that the spectrum of A consists of isolated eigenvalues of finite multiplicity satisfying (8.49). Thus, A satisfies the hypotheses of Theorems 8.7 and 8.9. Solutions of (8.11) satisfy (8.65). Hence, Theorems 8.7 and 8.9 produce weak solutions of (8.65). We summarize this as

Theorem 8.10. *Let $g(x)$ be a function satisfying the conditions described above. Then there exists a sequence of eigenvalues for the equation*

$$-\Delta u(x) + g(x)^2 u(x) = \lambda u(x), \quad x \in \mathbb{R}^n, \tag{8.66}$$

satisfying (8.49). Let $f(x,t)$ be a Carathéodory function satisfying hypothesis (A) for $\Omega = \mathbb{R}^n$, and assume that (8.47), (8.48), and (8.50) hold for some $\ell > 0$. Then (8.65) has at least one solution.

Remark 8.11. *We could have assumed*

$$a_1 \leq \liminf_{t\to-\infty} 2F(x,t)/t^2 \leq \limsup_{t\to-\infty} 2F(x,t)/t^2 \leq a_2$$

and

$$\gamma_\ell(a_1) \leq \liminf_{t\to\infty} 2F(x,t)/t^2 \leq \limsup_{t\to\infty} 2F(x,t)/t^2 \leq \Gamma_\ell(a_2)$$

in place of (8.50).

Remark 8.12. *This theorem generalizes results of several authors, including [10], [46], [71], [98], and [104], with various conditions on the function $g(x)$ to ensure that the spectrum of (8.66) is discrete. We guarantee it by assuming that multiplication by g^{-1} is a compact operator from $H^{1,2}$ to L^2. A sufficient condition for this is given in [106]. Since g^{-1} is bounded, a simple sufficient condition is that for each constant $b > 0$,*

$$m\{x \in \mathbb{R}^n : |x-y| < 1,\ g(x) < b\} \to 0 \text{ as } |y| \to \infty. \tag{8.67}$$

Remark 8.13. *If we choose $a_1 = \lambda_\ell$ and $a_2 = \lambda_{\ell+1}$, inequality (8.50) reduces to*

$$\lambda_\ell t^2 - W_1(x) \leq 2F(x,t) \leq \lambda_{\ell+1} t^2 + W_2(x), \quad x \in \mathbb{R}^n,\ t \in \mathbb{R}. \tag{8.68}$$

By choosing $a_1,\ a_2$ to be different values, we allow a wider range of possibilities for $F(x,t)$.

8.5 Further applications

Now we look for solutions of (8.13) under different conditions. Let A be a self-adjoint operator on $L^2(\Omega)$. We assume that $A \geq \lambda_0 > 0$ and that

$$C_0^\infty(\Omega) \subset D := D(A^{1/2}) \subset H^{m,2}(\Omega)$$

for some $m > 0$. Let q^* be given by

$$\begin{aligned} q^* &= 2n/(n-2m), && 2m < n \\ &= \infty, && n \leq 2m, \end{aligned}$$

and let $f(x,t)$ be a Carathéodory function on $\Omega \times \mathbb{R}$. We assume the following.

I. There are positive functions $V(x), V_0(x), V_1(x), W(x), W_1(x), r_0(t), r_1(t)$ and constants q, q_1 such that the following hold.

(a) $2 < q < q^*, 1 < q_1 < q^*$.

(b) Multiplication by V or W is a bounded operator from D to $L^2(\Omega)$, multiplication by V is compact from D to $L^{q-1}(K)$ for each compact subset K of Ω, and W is in $L^q(\Omega)$. Multiplication by V_0 is compact from D to $L^q(\Omega)$, V_0 is locally bounded, and

$$|f(x,st)| \leq V_0(x)(|V(x)s|^{q-1} + W(x)^{q-1})tr_0(t), \quad s \in \mathbb{R},\ t \geq 1. \tag{8.69}$$

(c) Multiplication by V_1 is a compact operator from D to $L^{q_1}(\Omega)$, W_1 is in $L^1(\Omega)$, and

$$|H(x, st)| \leq (|V_1(x)s|^{q_1} + W_1(x))r_1(t), \quad s \in \mathbb{R},\ t \geq 1. \tag{8.70}$$

(d) There is a $\psi \in \Psi$ such that

$$\psi(t) \leq tr_0(t), \quad t\psi(t) \leq r_1(t), \quad t \geq 1, \tag{8.71}$$

and

$$r_i(t) \to r_i \quad \text{as } t \to \infty, \quad i = 0, 1, \tag{8.72}$$

with

$$r_0 < \infty \tag{8.73}$$

and

$$r_1 \leq \infty. \tag{8.74}$$

II. There are positive constants α, β such that

$$2F(x, t) \leq \lambda_0(1 - \alpha)t^2, \quad t^2 < \beta. \tag{8.75}$$

III. (a) If $r_0 \neq 0$, it is assumed that there is a measurable function $\gamma(x, a)$ on $\Omega \times \mathbb{R}$ such that

$$f(x, st)/tr_0(t) \to \gamma(x, a) \text{ a.e.} \quad \text{as } s \to a,\ t \to \infty. \tag{8.76}$$

(b) If $r_0 \neq 0$ and $r_1 \neq 0$, it is assumed that there is a measurable function $\Gamma(x, a)$ on $\Omega \times \mathbb{R}$ such that

$$H(x, st)/r_1(t) \to \Gamma(x, a) \text{ a.e.} \quad \text{as } s \to a,\ t \to \infty. \tag{8.77}$$

If $r_1 \neq \infty$, then (8.77) is required to hold only for $a \neq 0$.

IV. If $r_0 \neq 0$ and $r_1 \neq 0$, then we assume that there does not exist a function $u \in D\backslash\{0\}$ and $b > 0$ such that

$$a(u) := \frac{1}{2}(Au, u) = 1, \tag{8.78}$$

$$r_0^{-1} Au = \gamma(x, u) \text{ a.e.}, \tag{8.79}$$

$$\int_\Omega \Gamma(x, u)\, dx = -b/r_1. \tag{8.80}$$

V. If $r_0 \neq 0$ and $r_1 \neq \infty$, we assume that all solutions $u \not\equiv 0$ of (8.79) are nonzero a.e. We have the following.

Theorem 8.14. *Under hypotheses I to V, there is a solution of*

$$Au = f(x,u) \tag{8.81}$$

in D. If there is a $u_0 \in D\backslash\{0\}$ such that

$$G(u_0) \le 0, \tag{8.82}$$

where

$$G(u) = a(u) - \int_\Omega F(x,u)\,dx, \tag{8.83}$$

then (8.81) has a nonzero solution.

Proof. Under hypothesis I(b), it is easily checked that $G(u)$ given by (8.83) is continuously Fréchet differentiable with

$$(G'(u),v) = 2a(u,v) - \int_\Omega f(x,u)v\,dx, \quad u,v \in D. \tag{8.84}$$

Also by hypothesis I(b),

$$\|Vu\|_q^2 + \|V_0u\|_q^2 + \|Wu\|_q^2 \le Ca(u), \quad u \in D. \tag{8.85}$$

Thus, by (8.75),

$$\begin{aligned} G(u) &\ge \alpha a(u) + (1-\alpha)a(u) - \frac{1}{2}\lambda_0(1-\alpha)\|u\|^2 \\ &\quad - C\int_{u^2>\beta} |V_0u|(|Vu|^{q-1} + \beta^{(1-q)/2}|Wu|^{q-1})\,dx \\ &\ge a(u)(\alpha C_1 a(u)^{(q/2)-1}). \end{aligned} \tag{8.86}$$

Hence, there are positive constants ρ, δ such that

$$G(u) \ge \rho, \quad a(u) = \delta^2. \tag{8.87}$$

We consider two cases.

Case 1. $G(u) \ge 0$ for all $u \in D$. In this case 0 is a minimum point and we must have $G'(0) = 0$. By (8.84), we see that 0 is a solution of (8.81).

Case 2. There is a $u_0 \in D\backslash\{0\}$ such that (8.82) holds. In this case the hypotheses of Theorem 2.18 are satisfied. In particular, there is a sequence $\{u_k\} \subset D$ such that

$$G(u_k) \to b \tag{8.88}$$

and

$$G'(u_k)/\psi(r_k) \to 0, \tag{8.89}$$

where b is given by (8.80), $t_k^2 = a(u_k)$, and $\psi(t) \in \Psi$ is the function satisfying (8.71). We shall show that this sequence satisfies

$$a(u_k) \le C. \tag{8.90}$$

If so, we can find a renamed subsequence that converges weakly in D to a function u and such that $V_0 u_k \to V_0 u$ in $L^q(\Omega)$ and a.e. in Ω while $V u_k \to V u$ in $L^{q-1}(K)$ for each compact subset K of Ω. By (8.84) and (8.89),

$$2a(u_k, v) - (f(x, u_k), v) \to 0, \quad v \in D. \tag{8.91}$$

If $v \in C_0^\infty(\Omega)$, then, by hypothesis I(b),

$$|f(x, u_k)v| \le C(|V(x)u_k| + W(x)^{q-1})|V_0 v|,$$

and the right-hand side converges in $L^1(\Omega)$ to

$$C(|Vu|^{q-1} + W^{q-1})|V_0 u|.$$

Thus,

$$(f(x, u_k), v) \to (f(x, u), v), \quad v \in C_0^\infty(\Omega). \tag{8.92}$$

Since $u_k \to u$ weakly in D, we have

$$2a(u, v) = (f(\cdot, u), v), \quad v \in C_0^\infty(\Omega), \tag{8.93}$$

which shows that u is a solution of (8.81).

I claim that $u \ne 0$. To see this, note that $b > 0$ by (8.87). Moreover, by (8.70),

$$|H(x, u_k)| \le C(|V_1 u_k|^{q_1} + W_1),$$

and the right-hand side converges in $L^1(\Omega)$ to

$$C(|V_1 u_k|^{q_1} + W_1).$$

Since

$$H(x, u_k) \to H(x, u) \text{ a.e.},$$

we have

$$\int_\Omega H(x, u_k)\, dx \to \int_\Omega H(x, u)\, dx.$$

But for any $u \in D$,

$$\frac{1}{2}(G'(u), u) = G(u) + \int_\Omega H(x, u)\, dx. \tag{8.94}$$

Hence, (8.88) and (8.89) imply that

$$\int_\Omega H(x, u)\, dx = -b \ne 0.$$

Thus, $u \ne 0$ since $H(x, 0) \equiv 0$.

It therefore remains only to prove (8.90). Assume that $t_k \to \infty$, and let $\tilde{u}_k = u_k/t_k$. Then

$$a(\tilde{u}_k) = 1 \tag{8.95}$$

and there is a renamed subsequence $\{\tilde{u}_k\}$ that converges weakly in D to a function $\tilde{u}$ and such that $V_0\tilde{u}_k \to V_0\tilde{u}$ in $L^q(\Omega)$ and a.e. in Ω while $Vu_k \to V\tilde{u}$ in $L^{q-1}(K)$ for each compact subset K of Ω. By (8.69),

$$|f(x, u_k)|/t_k r_0(t_k) \leq (|V\tilde{u}_k|^{q-1} + W^{q-1})|V_0\tilde{u}_k|.$$

Thus,

$$\left|\int_\Omega f(x, u_k)\,dx\right|/t_k r_0(t_k) \leq (\|V\tilde{u}_k\|_q^{q-1} + \|W\|_q^{q-1})\|V_0\tilde{u}_k\|_q. \tag{8.96}$$

Moreover, by (8.71) and (8.89),

$$2a(\tilde{u}_k)/r_0(t_k) - (f(x, u_k)/t_k r_0(t_k), \tilde{u}_k) \to 0. \tag{8.97}$$

This shows that $\tilde{u}(x) \equiv 0$. Otherwise, we would have $V_0\tilde{u}_k \to 0$ in $L^q(\Omega)$ and consequently the left-hand side of (8.96), which is equal to the second term of (8.97), will converge to 0. But this would mean that $a(\tilde{u}_k) \to 0$, contradicting (8.95).

If $r_0 = 0$, we obtain another contradiction, for the second term in (8.97) is bounded by (8.96) while the first becomes infinite by (8.95) if $t_k \to \infty$. Hence, $r_0 = 0$ implies (8.90), and the proof is complete in this case.

It remains to consider the case when $r_0 \neq 0$. By (8.69),

$$|f(x, u_k)v|/t_k r_0(t_k) \leq (|V\tilde{u}_k|^{q-1} + W^{q-1})|V_0 v|.$$

When $v \in C_0^\infty(\Omega)$, the right-hand side converges in $L^1(\Omega)$ to

$$(|V\tilde{u}|^{q-1} + W^{q-1})|V_0 v|$$

by hypothesis II(b). By (8.76), the left-hand side converges a.e. to $\gamma(x, \tilde{u})$. Thus,

$$\int_\Omega f(x, u_k)v(x)dx/t_k r_0(t_k) \to \int_\Omega \gamma(x, \tilde{u})v(x)\,dx \tag{8.98}$$

for each $v \in C_0^\infty(\Omega)$. By (8.71) and (8.89),

$$2a(\tilde{u}_k/t_0(t_k), v) - (f(x, u_k)/t_k r_0(t_k), v) \to 0 \tag{8.99}$$

for each $v \in D$. Thus, by (8.72) and (8.98),

$$2a(\tilde{u}/r_0, v) = (\gamma(x, u), v), \quad v \in C_0^\infty(\Omega).$$

This shows that $\tilde{u}$ is a solution of (8.79). By (8.70),

$$|H(x, u_k)|/r_1(t_k) \leq |V_1\tilde{u}_k|^{q_1} + W_1, \tag{8.100}$$

and the right-hand side converges in $L^1(\Omega)$ to the function

$$|V_1\tilde{u}|^{q_1} + W_1$$

by hypothesis II(c). If $r_1 = 0$, this produces a contradiction, for by (8.88), (8.89), and (8.94),

$$\int_\Omega H(x, u_k)\,dx / r_1(t_k) \tag{8.101}$$

converges to $-b/r_1$. Since $b \neq 0$, this means that expression (8.101) becomes infinite. Thus, the proof is complete for the case when $r_1 = 0$.

It therefore remains to consider the case when $r_0 \neq 0,\ r_1 \neq 0$. If $r_1 = \infty$, the integrand of (8.101) converges a.e. to $\Gamma(x, \tilde{u})$ by hypothesis III(b). If $r_1 \neq \infty$, this convergence takes place only for those points x where $\tilde{u}(x) \neq 0$. But by hypothesis V, $\tilde{u}(x) \neq 0$ a.e. since it is a solution of (8.79). Hence, the convergence is a.e. in all cases, and expression (8.101) converges to

$$\int_\Omega \Gamma(x, \tilde{u})\,dx. \tag{8.102}$$

Since it also converges to $-b/r_1$ by (8.88), (8.89) and (8.94), we see that $\tilde{u}$ is a solution of (8.80) as well as (8.79). As before, (8.97) holds by (8.71) and (8.89). In view of (8.76), this implies that

$$2/r_0 = (\gamma(x, \tilde{u}), \tilde{u}).$$

If we combine this with (8.79), we see that $\tilde{u}$ satisfies (8.78) as well as (8.79) and (8.80). A contradiction is now provided by hypothesis IV. Hence, (8.90) holds, and the proof is complete. □

8.6 Special cases

In this section we present some consequences of Theorem 8.14.

Theorem 8.15. *Assume that*

$$|H(x, t)| \leq W(x) \in L^1(\Omega), \tag{8.103}$$

$$H(x, t) \to H_\pm(x) \text{ as } t \to \pm\infty \text{ a.e.}, \tag{8.104}$$

$$2t^{-2}F(x, t) \to b_\pm(x) \text{ as } t \to \pm\infty \text{ a.e.}, \tag{8.105}$$

$$|f(x, t)| \leq W_1(x) \in L^1(\Omega), \quad |t| \leq 1, \tag{8.106}$$

and that

$$\int_{u>0} H_+(x)\,dx + \int_{u<0} H_-(x)\,dx \geq 0 \tag{8.107}$$

for all $u \in D\backslash\{0\}$ satisfying

$$Au = b_+(x)u^+ - b_-(x)u^-, \tag{8.108}$$

where $u^{\pm}(x) = \max[\pm u(x), 0]$. *Assume that there is a* $q > 2$ *such that* $b^{1/q}$, $W^{1/q}$ *are compact operators from* D *to* $L^q(\Omega)$, *where* $b(x) = \max |b_{\pm}(x)|$, *and that solutions of (8.108) that are not identically zero a.e. are never zero a.e. Then (8.81) has a solution* $u \in D\backslash\{0\}$.

Theorem 8.16. *If we replace (8.103), (8.104) in Theorem 8.15 by*

$$|H(x,t)| \le V(x)|t|^{\gamma} + W(x), \quad 0 < \gamma < 2, \tag{8.109}$$

and

$$H(x,t)/|t|^{\gamma} \to H_{\pm}(x) \text{ as } t \to \pm\infty \text{ a.e.}, \tag{8.110}$$

the theorem will hold if

$$\int_{u>0} H_{\pm}(x)|u|^{\gamma}\,dx + \int_{u<0} H_{-}(x)|u|^{\gamma}\,dx > 0 \tag{8.111}$$

for all $u \in D\backslash\{0\}$ *satisfying (8.108), and* $V^{1/q}$ *is a compact operator from* D *to* $L^q(\Omega)$. *In this case the requirement that solutions of (8.108) that are not identically zero a.e. are never zero a.e. can be removed.*

Theorem 8.17. *Assume that there is a number* γ *such that* $0 \le \gamma < 1$ *and*

$$|f(x,t)| \le V(x)|t|^{\gamma} + W(x), \tag{8.112}$$

with $V^{1/q}$, $W^{1/q}$ *compact operators from* D *to* $L^q(\Omega)$, *where* $q > 2$. *Then (8.81) has a solution* $u \in D\backslash\{0\}$.

Theorem 8.18. *Assume that*

$$|f(x,t)| \le V(x)|t| + W(x) \tag{8.113}$$

and

$$f(x,t)/t \to b_{\pm}(x) \text{ as } t \to \pm\infty \text{ a.e.}, \tag{8.114}$$

with $b^{1/q}$, $V^{1/q}$, *and* $W^{1/q}$ *compact operators from* D *to* $L^q(\Omega)$, *where* $b(x) = \max |b \pm (x)|$, *and* $q > 2$. *Assume in addition that (8.108) has no nontrivial solutions. Then (8.81) has a solution* $u \in D\backslash\{0\}$.

8.7 The proofs

In this section we give the proofs of Theorems 8.15 to 8.18. First, we give the

Proof of Theorem 8.15. We apply Theorem 8.14. We take $r_0(t) = r_1(t) = 1$, $\psi(t) = t^{-1}$. Inequality (8.70) is satisfied with $q_1 = q$, V_1 arbitrary, and $W_1 = W$. Since

$$\partial(Ft^{-2})/\partial t = -2t^{-3}H(x,t), \tag{8.115}$$

we have

$$F(x,t) = \begin{cases} \frac{1}{2}b_+(x)t^2 + F_0(x,t), & t > 0, \\ \frac{1}{2}b_-(x)t^2 + F_0(x,t), & t < 0, \end{cases} \tag{8.116}$$

where

$$F_0(x,t) = \begin{cases} 2t^2 \int_t^\infty s^{-3}H(x,s)\,ds, & t > 0, \\ -2t^2 \int_{-\infty}^t s^{-3}H(x,s)\,ds, & t < 0. \end{cases} \tag{8.117}$$

Thus,

$$f(x,t) = \begin{cases} b_+(x)t + 4t(\int_t^\infty) - 2H(x,t)/t, & t > 0, \\ b_-(x)t - 4t(\int_{-\infty}^t) - 2H(x,t)/t & t < 0. \end{cases} \tag{8.118}$$

Consequently,

$$|f(x,t)| \leq b(x)|t| + 4W|t|^{-1}, \quad |t| \geq 1. \tag{8.119}$$

This combined with (8.106) gives (8.69) with $V_0 = V = b^{1/q}$. Moreover, by (8.119),

$$f(x,st)/t \to \pm b^{\pm} \quad \text{as } s \to a,\ t \to \infty. \tag{8.120}$$

Thus, (8.79) becomes (8.108). Moreover, by (8.104), the left-hand side of (8.80) is the left-hand side of (8.107). Hence, (8.107) assures us that (8.80) cannot hold for any solution $u \in D$ of (8.79). Thus, all of the hypotheses of Theorem 8.14 are satisfied. □

Proof of Theorem 8.16. In this case we take $r_0(t) = 1,\ r_1(t) = t^\gamma$. Since $r_1 = \infty$, the right-hand side of (8.80) vanishes. Thus, strict inequality in (8.111) is needed to guarantee that no solutions of (8.78)–(8.80) exist. Moreover, we do not require hypothesis V for this case. □

Proof of Theorem 8.17. Here we take $r_0(t) = t^{\gamma-1}$ and $r_1(t) = t^{\gamma+1}$. Note that by (8.112),

$$|F(x,t)| \leq p^{-1}V(x)|t|^p + W(x)|t|, \tag{8.121}$$

where $p = 1 + \gamma$. Thus, by (8.2),

$$|H(x,t)| \leq \left(\frac{1}{2} + \frac{1}{p}\right) V(x)|t|^p + \frac{3}{2}W(x)|t|. \tag{8.122}$$

Since $r_0 = 0$ and $r_1 = \infty$, all of the hypotheses of Theorem 8.14 hold. □

Proof of Theorem 8.18. Now we take $r_0(t) = 1,\ r_1(t) = t^2$. By (8.113),

$$|F(x,t)| \leq \frac{1}{2}V(x)t^2 + W(x)|t|$$

and

$$|H(x,t)| \le \frac{1}{2}V(x)t^2 + \frac{3}{2}W(x)|t|, \tag{8.123}$$

Moreover, by (8.114),

$$2F(x,t)/t^2 \to b_\pm(x) \text{ a.e. as } t \to \pm\infty. \tag{8.124}$$

Consequently, by (8.2),

$$H(x,t)/t^2 \to 0 \text{ a.e. as } t \to \pm\infty.$$

Thus $r'(x,a) \to 0$ a.e. and (8.80) is always satisfied. Thus, we must assume that (8.79) has no solutions in $D\backslash\{0\}$. $\square$

8.8 Notes and remarks

The material in this chapter is from [120], [110], [107], and [132].

Chapter 9

Superlinear Problems

9.1 Introduction

Consider the problem

(9.1) $$-\Delta u = f(x,u),\ x \in \Omega; \quad u = 0 \ \text{ on } \ \partial\Omega,$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain whose boundary is a smooth manifold, and $f(x,t)$ is a continuous function on $\bar{\Omega} \times \mathbb{R}$. This semilinear Dirichlet problem has been studied by many authors. It is called **sublinear** if there is a constant C such that

$$|f(x,t)| \le C(|t|+1), \quad x \in \Omega,\ t \in \mathbb{R}.$$

Otherwise, it is called **superlinear.** Beginning in [7], almost all researchers studying the superlinear problem assumed

(a_1) There are constants $c_1,\ c_2 \ge 0$ such that

$$|f(x,t)| \le c_1 + c_2|t|^s,$$

where $0 \le s < (n+2)/(n-2)$ if $n > 2$.

(a_2) $f(x,t) = o(|t|)$ as $t \to 0$.

(a_3) There are constants $\mu > 2,\ r \ge 0$ such that

(9.2) $$0 < \mu F(x,t) \le t f(x,t), \quad |t| \ge r,$$

where

$$F(x,t) = \int_0^t f(x,s)\,ds.$$

They proved

Theorem 9.1. *Under hypotheses* (a_1)–(a_3), *problem (9.1) has a nontrivial weak solution.*

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The condition (a_3) is convenient, but it is very restrictive. In particular, it implies that there exist positive constants $c_3,\ c_4$ such that

$$F(x,t) \geq c_3|t|^\mu - c_4, \quad x \in \Omega,\ t \in \mathbb{R}. \tag{9.3}$$

Although this condition is weaker, it still eliminates many superlinear problems.

A much weaker condition that implies superlinearity is

(a_3') Either

$$F(x,t)/t^2 \to \infty \text{ as } t \to \infty$$

or

$$F(x,t)/t^2 \to \infty \text{ as } t \to -\infty.$$

The purpose of the present chapter is to explore what happens when (a_3) is replaced with (a_3'). Surprisingly, we find the following to be true.

Theorem 9.2. *Under hypotheses* (a_1), (a_2), (a_3'), *the boundary-value problem*

$$-\Delta u = \beta f(x,u),\ x \in \Omega; \quad u = 0 \text{ on } \partial\Omega, \tag{9.4}$$

has a nontrivial solution for almost every positive β.

We generalize this theorem and present some variations below.

9.2 The main theorems

We consider the boundary-value problems described in Sections 8.2 and 8.4 under assumption **(A)** stipulated there. We allow the domain Ω to be unbounded. We also assume

(B) The point λ_0 is an isolated simple eigenvalue with a bounded eigenfunction $\varphi_0(x) \neq 0$ a.e. in Ω.

(C) There is a $\delta > 0$ such that

$$2F(x,t) \leq \lambda_0 t^2, \quad |t| \leq \delta,\ \ x \in \Omega,$$

where

$$F(x,t) := \int_0^t f(x,s)ds. \tag{9.5}$$

(D) There is a function $W(x) \in L^1(\Omega)$ such that either

$$W(x) \leq F(x,t)/t^2 \to \infty \text{ as } t \to \infty, \quad x \in \Omega$$

or

$$W(x) \leq F(x,t)/t^2 \to \infty \text{ as } t \to -\infty, \quad x \in \Omega.$$

(The function $W(x)$ need not be positive.)

(E) There are constants $\mu > 2, C \geq 0$ such that

$$[\mu F(x,t) - tf(x,t)]/(t^2+1) \leq C, \quad t \in \mathbb{R},\ \ x \in \Omega.$$

We shall prove

Theorem 9.3. *Under the above hypotheses, the problem*

$$Au = f(x, u), \quad u \in D \tag{9.6}$$

has at least one nontrivial solution.

We also have

Theorem 9.4. *If we replace hypothesis* **(E)** *with*
(E′) *The function*

$$H(x, t) := tf(x, t) - 2F(x, t) \tag{9.7}$$

is convex in t.
then problem (9.6) has at least one nontrivial solution.

As we noted, problem (9.6) is called sublinear if $f(x, t)$ satisfies

$$|f(x, t)| \leq C(|t| + 1), \quad x \in \Omega, \ t \in R. \tag{9.8}$$

Otherwise, it is called superlinear. Hypothesis **(D)** requires (9.6) to be superlinear.

If we drop hypothesis **(E)** completely, then we are able to prove the following theorems.

Theorem 9.5. *If we replace hypotheses* **(C), (D)** *with*
(C′) *There are a $\delta > 0$ and a $\tilde{\lambda} > \lambda_0$ such that*

$$2F(x, t) \geq \tilde{\lambda} t^2, \quad |t| \leq \delta, \ x \in \Omega$$

and
(D′) *there is a function $W(x) \in L^1(\Omega)$ such that*

$$W(x) \geq P(x, t) \to -\infty \ \text{ as } \ |t| \to \infty, \quad x \in \Omega,$$

where

$$P(x, t) := F(x, t) - \frac{1}{2}\lambda_0 t^2. \tag{9.9}$$

and drop hypothesis **(E)**, *then problem (9.6) has at least one nontrivial solution.*

We also have

Theorem 9.6. *Assume that* **(A)–(D)** *hold. Then, for almost every $\beta \in (0, 1)$, the equation*

$$Au = \beta f(x, u) \tag{9.10}$$

has a nontrivial solution. In particular, the eigenvalue problem (9.10) has infinitely many solutions.

Theorem 9.7. *If we replace hypothesis* **(C)** *in Theorem 9.6 with*
(C″) *There are a $\delta > 0$ and a $\tilde{\lambda} \le \lambda_0$ such that*

$$2F(x,t) \le \tilde{\lambda} t^2, \quad |t| \le \delta,\ x \in \Omega.$$

and **(D)** *with*
(D″) *Either*

$$\int_\Omega F(x, R\varphi_0)\, dx/R^2 \to \infty \text{ as } R \to \infty$$

or

$$\int_\Omega F(x, -R\varphi_0)\, dx/R^2 \to \infty \text{ as } R \to \infty.$$

then (9.10) has a nontrivial solution for almost every $\beta \in (0, \lambda_0/\tilde{\lambda})$.

Corollary 9.8. *If we replace hypothesis* **(C″)** *in Theorem 9.7 with*
(C‴) $F(x,t)/t^2 \to 0$ uniformly as $t \to 0$.
then (9.10) has a nontrivial solution for almost every $\beta \in (0, \infty)$.

9.3 Preliminaries

Define

$$G(u) := \|u\|_D^2 - 2\int_\Omega F(x,u)dx. \tag{9.11}$$

Under hypothesis **(A)**, it is known that G is a continuously differentiable functional on the whole of D. In fact, the following were proved in [122, pp. 56–58].

Proposition 9.9. *Under hypothesis* **(A)**, *$F(x,u(x))$ and $v(x)f(x,u(x))$ are in $L^1(\Omega)$ whenever $u,\ v \in D$.*

Proposition 9.10. *$G(u)$ has a Fréchet derivative $G'(u)$ on D given by*

$$(G'(u), v)_D = 2(u,v)_D - 2(f(\cdot,u), v). \tag{9.12}$$

Proposition 9.11. *The derivative $G'(u)$ given by (9.12) is continuous in u.*

Theorem 9.12. *Under hypotheses* **(A)–(C)**, *the following alternative holds:*
Either

(a) *there is an infinite number of $y(x) \in D(A) \setminus \{0\}$ such that*

$$Ay = f(x,y) = \lambda_0 y \tag{9.13}$$

or

(b) *for each $\rho > 0$ sufficiently small, there is an $\varepsilon > 0$ such that*

$$G(u) \ge \varepsilon, \quad \|u\|_D = \rho. \tag{9.14}$$

9.4 Proofs

We now give the proof of Theorem 9.3.

Proof. We take

$$G(u) = \|u\|_D^2 - 2\int_\Omega F(x,u)dx. \tag{9.15}$$

Under our hypotheses, Propositions 9.9–9.11 apply, and

$$(G'(u), v) = 2(u,v)_D - 2(f(\cdot,u),v), \quad u, v \in D. \tag{9.16}$$

By Theorem 9.12, we see that there are positive constants ε, ρ such that

$$G(u) \geq \varepsilon, \quad \|u\|_D = \rho, \tag{9.17}$$

unless

$$Au = \lambda_0 u = f(x,u), \quad u \in D \setminus \{0\}, \tag{9.18}$$

has a solution. This would give a nontrivial solution of (9.6). We may therefore assume that (9.17) holds. Next, we note that

$$G(\pm R\varphi_0)/R^2 = \|\varphi_0\|_D^2 - 2\int_\Omega \{F(x, \pm R\varphi_0)/R^2\varphi_0^2\}\varphi_0^2 dx \to -\infty \text{ as } R \to \infty$$

by hypothesis **(D)**, depending on which part of **(D)** is assumed, since $\varphi_0 \neq 0$ a.e. Since $G(0) = 0$ and (9.17) holds, we can now apply Theorem 3.13 to conclude that there is a sequence $\{u_k\} \subset D$ such that

$$G(u_k) \to c \geq \varepsilon, \quad G'(u_k) \to 0.$$

Then

$$G(u_k) = \rho_k^2 - 2\int F(x,u_k)dx \to c \tag{9.19}$$

and

$$(G'(u_k), u_k) = 2\rho_k^2 - 2(f(\cdot,u_k),u_k) = o(\rho_k), \tag{9.20}$$

where $\rho_k = \|u_k\|_D$. Assume that $\rho_k \to \infty$, and let $\tilde{u}_k = u_k/\rho_k$. Since $\|\tilde{u}_k\|_D = 1$, there is a renamed subsequence such that $\tilde{u}_k \to \tilde{u}$ weakly in D, strongly in $L^2(\Omega)$, and a.e. in Ω. By (9.19),

$$\int_\Omega \frac{2F(x,u_k)}{u_k^2}\tilde{u}_k^2\, dx \to 1.$$

Let

$$\Omega_1 = \{x \in \Omega : \tilde{u}(x) \neq 0\}, \quad \Omega_2 = \Omega \setminus \Omega_1.$$

Then

$$\frac{2F(x,u_k)}{u_k^2}\tilde{u}_k^2 \to \infty, \quad x \in \Omega_1,$$

by hypothesis **(D)**. If Ω_1 has positive measure, then

$$\int_\Omega \frac{2F(x,u_k)}{u_k^2}\tilde{u}_k^2\,dx \geq \int_{\Omega_1} \frac{2F(x,u_k)}{u_k^2}\tilde{u}_k^2\,dx + \int_{\Omega_2}[-W(x)]\,dx \to \infty.$$

Thus, the measure of Ω_1 must be 0, i.e., we must have $\tilde{u} \equiv 0$ a.e. Moreover,

$$\int_\Omega \frac{2F(x,u_k) - \mu u_k f(x,u_k)}{u_k^2}\tilde{u}_k^2\,dx \to 1-\mu.$$

But by hypothesis **(E)**,

$$\limsup \frac{2F(x,u_k) - \mu u_k f(x,u_k)}{u_k^2}\tilde{u}_k^2 \leq \limsup C\frac{u_k^2+1}{u_k^2}\tilde{u}_k^2 = 0,$$

which implies that $1-\mu \leq 0$, contrary to our assumption. Hence, the ρ_k are bounded. We can now follow the usual procedures to obtain a weak solution of (9.6) satisfying $G(u) = c \geq \varepsilon$ (cf., e.g., [122, p. 64]). Since $G(0) = 0$, we see that $u \neq 0$. This completes the proof. □

We postpone the proof of Theorem 9.4 until the next section.

In proving Theorem 9.5, we shall make use of

Lemma 9.13. *Under hypothesis* **(C′)***, there is an* $\alpha \neq 0$ *such that* $G(\alpha\varphi_0) < 0$.

Proof. We can assume that

$$\|\varphi_0\|_D = 1. \tag{9.21}$$

Thus,

$$\begin{aligned} G(\alpha\varphi_0) &= \alpha^2 - 2\int_\Omega F(x,\alpha\varphi_0)\,dx \\ &\leq \alpha^2 - \tilde{\lambda}\alpha^2 \int_{|\alpha\varphi_0(x)|<\delta} \varphi_0(x)^2 dx \\ &\quad + \int_{|\alpha\varphi_0(x)|>\delta} V^q(|\alpha\varphi_0|^q + |\alpha\varphi_0|) \\ &\leq \alpha^2 - \tilde{\lambda}\alpha^2\|\varphi_0\|^2 + C|\alpha|^q\|V\varphi_0\|_q^q \\ &\leq \alpha^2[1 - (\tilde{\lambda}/\lambda_0) + C'|\alpha|^{q-2}]. \end{aligned}$$

This can be made negative by taking α sufficiently small. □

Lemma 9.14. *Under hypothesis* **(D′)***,*

$$G(u) \to \infty \quad \text{as} \quad \|u\|_D \to \infty. \tag{9.22}$$

Proof. Suppose there is a sequence $\{u_k\} \subset D$ such that $\rho_k = \|u_k\| \to \infty$ and

$$G(u_k) \leq K.$$

Write

$$u_k = w_k + \alpha_k \varphi_0, \quad \tilde{u}_k = u_k/\rho_k, \quad \tilde{w}_k = w_k/\rho_k, \quad \tilde{\alpha}_k = \alpha_k/\rho_k,$$

where $w_k \perp \varphi_0$. If $\lambda_1 > \lambda_0$ is the next point in the spectrum of A, then

$$\lambda_1 \|w\|^2 \leq \|w\|_D^2, \quad w \perp \varphi_0.$$

Thus,

$$\begin{aligned} G(u_k) &= \|u_k\|_D^2 - \lambda_0 \|u_k\|^2 - 2\int_\Omega P(x, u_k)\, dx \\ &\geq \left(1 - \frac{\lambda_0}{\lambda_1}\right) \|w_k\|_D^2 - 2\int_\Omega P(x, u_k)\, dx \\ &\geq \left(1 - \frac{\lambda_0}{\lambda_1}\right) \|w_k\|_D^2 - 2\int_\Omega W(x)\, dx. \end{aligned}$$

The only way this would not converge to ∞ is if $\|w_k\|_D$ is bounded. But then $\|\tilde{w}_k\|_D \to 0$ and $|\tilde{\alpha}_k| \to 1$. Since $\|\tilde{u}_k\|_D = 1$, there is a renamed subsequence such that $\tilde{u}_k \to \tilde{u}$ weakly in D, strongly in $L^2(\Omega)$, and a.e. in Ω. Since $\tilde{w} = 0$ and $|\tilde{\alpha}| = 1$, we have $\tilde{u}(x) = \tilde{\alpha}\varphi_0(x) \neq 0$ a.e. Hence, $|u_k(x)| = \rho_k |\tilde{u}_k(x)| \to \infty$ a.e. Consequently,

$$\int_\Omega P(x, u_k)\, dx \to -\infty,$$

showing that $G(u_k) \to \infty$. This completes the proof. □

We can now give the proof of Theorem 9.5.

Proof. Let

$$m = \inf_D G.$$

Then there is a sequence $\{u_k\} \subset D$ such that $G(u_k) \to m$. In view of Lemma 9.14, we must have $\|u_k\|_D \leq C$. Thus, there is a renamed subsequence such that $u_k \to u$ weakly in D, strongly in $L^2_{\text{loc}}(\Omega)$, and a.e. in Ω. Now,

$$\begin{aligned} G(u) &= \|u\|_D^2 - 2\int_\Omega F(x, u)\, dx \\ &= \|u_k\|_D^2 - 2([u_k - u], u)_D - \|u_k - u\|_D^2 \\ &\quad - 2\int_\Omega F(x, u_k)\, dx + 2\int_\Omega [F(x, u_k) - F(x, u)] dx \\ &\leq G(u_k) - 2([u_k - u], u)_D + 2\int_\Omega [F(x, u_k) - F(x, u)] dx. \end{aligned}$$

From our hypotheses, it follows that

$$\int_\Omega F(x, u_k)\,dx \to \int_\Omega F(x, u)\,dx$$

(cf., e.g., [122, p. 64]). We therefore have in the limit $G(u) \le m$, from which we conclude that $G(u) = m$ and $G'(u) = 0$. Hence, u is a weak solution of (9.6). We see from Lemma 9.13 that $m < 0$. Since $G(0) = 0$, we see that $u \neq 0$. This completes the proof. □

9.5 The parameter problem

In this section we shall give the proofs of Theorems 9.4, 9.6, and 9.7. They will be based on the following result proved in the next section. Let E be a reflexive Banach space with norm $\|\cdot\|$, and let A, B be two closed subsets of E. Suppose that $G \in C^1(E, \mathbb{R})$ is of the form $G(u) := I(u) - J(u), u \in E$, where $I, J \in C^1(E, \mathbb{R})$ map bounded sets to bounded sets. Define

$$G_\lambda(u) = \lambda I(u) - J(u), \quad \lambda \in \Lambda,$$

where Λ is an open interval contained in $(0, +\infty)$. Assume one of the following alternatives holds.

($\mathbf{H_1}$) $I(u) \ge 0$ for all $u \in E$ and either $I(u) \to \infty$ or $|J(u)| \to \infty$ as $\|u\| \to \infty$.

($\mathbf{H_2}$) $I(u) \le 0$ for all $u \in E$ and either $I(u) \to -\infty$ or $|J(u)| \to \infty$ as $\|u\| \to \infty$.

Furthermore, we suppose that

($\mathbf{H_3}$) $a_0(\lambda) := \sup_A G_\lambda \le b_0(\lambda) := \inf_B G_\lambda$, for any $\lambda \in \Lambda$.

We let Φ be the set of mappings $\Gamma(t) \in C(E \times [0, 1], E)$ described in Chapter 3.

We have

Theorem 9.15. *Assume that* ($\mathbf{H_1}$) *or* ($\mathbf{H_2}$) *holds together with* ($\mathbf{H_3}$).

(1) *If A links B [hm] and A is bounded, then, for almost all $\lambda \in \Lambda$, there exists $u_k(\lambda) \in E$ such that* $\sup_k \|u_k(\lambda)\| < \infty$, $G'_\lambda(u_k(\lambda)) \to 0$, *and*

$$G_\lambda(u_k(\lambda)) \to a(\lambda) := \inf_{\Gamma\in\Phi} \sup_{s\in[0,1], u\in A} G_\lambda(\Gamma(s)u), \quad k \to \infty.$$

Furthermore, if $a(\lambda) = b_0(\lambda)$, *then* $\operatorname{dist}(u_k(\lambda), B) \to 0,\ k \to \infty$.

(2) *If B links A[hm] and B is bounded, then, for almost all $\lambda \in \Lambda$, there exists $v_k(\lambda) \in E$ such that* $\sup_k \|v_k(\lambda)\| < \infty$, $G'_\lambda(v_k(\lambda)) \to 0$, *and*

$$G_\lambda(v_k(\lambda)) \to b(\lambda) := \sup_{\Gamma\in\Phi} \inf_{s\in[0,1], v\in B} G_\lambda(\Gamma(s, v)), \quad k \to \infty.$$

Furthermore, if $a_0(\lambda) = b(\lambda)$, *then* $\operatorname{dist}(v_k(\lambda), A) \to 0,\ k \to \infty$.

We shall also need the following extension of Theorem 9.12.

Theorem 9.16. *Let λ be a parameter satisfying $1 < \lambda \le K < \infty$. Under hypotheses* **(A)–(D)**, *for each $\rho > 0$ sufficiently small (not depending on λ), we have*

$$G_\lambda(u) := \lambda\|u\|_D^2 - 2\int_\Omega F(x,u)dx \ge (\lambda-1)\rho^2, \quad \|u\|_D = \rho. \tag{9.23}$$

If we replace hypothesis **(C)** *with hypothesis* **(C″)**, *assuming $1 < \tilde{\lambda}/\lambda_0 < \lambda \le K < \infty$, then we have*

$$G_\lambda(u) \ge \left(\lambda - \frac{\tilde{\lambda}}{\lambda_0}\right)\rho^2, \quad \|u\|_D = \rho. \tag{9.24}$$

Proof. Let $\lambda_1 > \lambda_0$ be the next point in the spectrum of A, and let N_0 denote the eigenspace of λ_0. We take $M = N_0^\perp \cap D$. By hypothesis **(B)**, there is a $\rho > 0$ such that

$$\|y\|_D \le \rho \Rightarrow |y(x)| \le \delta/2, \quad y \in N_0.$$

Now suppose $u \in D$ satisfies

$$\|u\|_D \le \rho \quad \text{and} \quad |u(x)| \ge \delta \tag{9.25}$$

for some $x \in \Omega$. We write

$$u = w + y, \quad w \in M, \quad y \in N_0. \tag{9.26}$$

Then for those $x \in \Omega$ satisfying (9.25) we have

$$\delta \le |u(x)| \le |w(x)| + |y(x)| \le |w(x)| + (\delta/2).$$

Hence,

$$|y(x)| \le \delta/2 \le |w(x)|, \tag{9.27}$$

and consequently,

$$|u(x)| \le 2|w(x)| \tag{9.28}$$

for all such x. Now we have by (9.26) and (9.28),

$$\begin{aligned}
G_\lambda(u) &\ge \lambda\|u\|_D^2 - \lambda_0\int_{|u|<\delta} u^2dx - C\int_{|u|>\delta}(|Vu|^q + V^q|u|)dx \\
&\ge \lambda\|u\|_D^2 - \lambda_0\|u\|^2 - C'\int_{|u|>\delta}|Vu|^q dx \\
&\ge (\lambda-1)\|y\|_D^2 + \lambda\|w\|_D^2 - \lambda_0\|w\|^2 - C''\int_{2|w|>\delta}|Vw|^q dx
\end{aligned}$$

in view of the fact that $\|y\|_D^2 = \lambda_0 \|y\|^2$ and (9.28) holds. Thus, by (8.3),

$$G_\lambda(u) \geq (\lambda - 1)\|y\|_D^2 + \left(\lambda - \frac{\lambda_0}{\lambda_1} - C'''\|w\|_D^{q-2}\right)\|w\|_D^2, \quad \|u\|_D \leq \rho. \tag{9.29}$$

We take $\rho > 0$ to satisfy

$$1 - \frac{\lambda_0}{\lambda_1} > C'''\rho^{q-2}.$$

This gives

$$\begin{aligned} G_\lambda(u) &\geq (\lambda - 1)\rho^2 + \left(\lambda - \frac{\lambda_0}{\lambda_1} - C'''\rho^{q-2} - \lambda + 1\right)\|w\|_D^2 \\ &\geq (\lambda - 1)\rho^2, \quad \|u\|_D = \rho. \end{aligned}$$

Hence, (9.23) holds.

To prove (9.24) under hypothesis **(C$''$)**, let $\sigma = \tilde{\lambda}/\lambda_0$ and $\Lambda = (\sigma, K)$. Under hypothesis **(C$''$)**, we have in place of (9.29)

$$G_\lambda(u) \geq (\lambda - \sigma)\|y\|_D^2 + \left(\lambda - \frac{\tilde{\lambda}}{\lambda_1} - C'''\|w\|_D^{q-2}\right)\|w\|_D^2, \quad \|u\|_D \leq \rho. \tag{9.30}$$

We take $\rho > 0$ to satisfy

$$\sigma - \frac{\tilde{\lambda}}{\lambda_1} > C'''\rho^{q-2}.$$

Consequently,

$$\begin{aligned} G_\lambda(u) &\geq (\lambda - \sigma)\rho^2 + \left(\lambda - \frac{\tilde{\lambda}}{\lambda_1} - C'''\rho^{q-2} - \lambda + \sigma\right)\|w\|_D^2 \\ &\geq (\lambda - \sigma)\rho^2, \quad \|u\|_D = \rho. \end{aligned}$$

This gives (9.24), and the proof is complete. □

We now turn to the proofs of Theorems 9.6 and 9.7. We prove the latter first. We shall prove Theorem 9.7 by applying Theorems 9.15 and 9.16.

Proof. We take $E = D$, $\Lambda = (\sigma, K)$, where $\sigma = \tilde{\lambda}/\lambda_0$, $K > 1$ is a finite number, and

$$I(u) = \|u\|_D^2, \quad J(u) = 2\int_\Omega F(x, u)\,dx.$$

For the purpose of this application, it is sufficient to know that the sets

$$A_\pm = [0, \pm R\varphi_0], \quad B = \{x \in D : \|x\|_D = \rho\}$$

link each other if $R > \rho$ (cf., e.g., [120]). In our case hypothesis **(H$_1$)** is satisfied. We now check that **(H$_3$)** holds. We observe that

$$G'_\lambda(u) = 0$$

is equivalent to (9.10) with $\beta = 1/\lambda$. Now, at least one of the expressions

$$J(\pm R\varphi_0)/R^2 = 2\int_\Omega F(x, \pm R\varphi_0)\, dx/R^2 \to \infty \text{ as } R \to \infty$$

by hypothesis $(\mathbf{D}'')$. Hence, for R sufficiently large, one of the inequalities

$$G_\lambda(\pm R\varphi_0)/R^2 \leq K\|\varphi_0\|_D^2 - 2\int_\Omega \{F(x, \pm R\varphi_0)/R^2 \leq 0$$

holds. Thus,

$$a_0(\lambda) \leq 0, \quad \lambda \in \Lambda.$$

Moreover, it follows from Theorem 9.16 that (9.24) holds. Hence,

$$b_0(\lambda) \geq (\lambda - \sigma)\rho^2, \quad \lambda \in \Lambda.$$

This shows that hypothesis $(\mathbf{H_3})$ holds. We can now apply Theorem 9.15 to conclude that for almost all $\lambda \in \Lambda$, there exists $u_k(\lambda) \in D$ such that $\sup_k \|u_k(\lambda)\| < \infty$, $G'_\lambda(u_k(\lambda)) \to 0$, and

$$G_\lambda(u_k(\lambda)) \to a(\lambda) \geq b_0(\lambda).$$

Once it is known that the sequence $\{u_k\}$ is bounded, we can apply the usual theory to conclude that there is a solution of

$$G'_\lambda(u) = 0, \quad G_\lambda(u) = a(\lambda)$$

(cf., e.g., [122, p. 64]). Moreover, from the definition, we see that $a(\lambda) \geq (\lambda - \sigma)\rho^2$. Hence, the equation $G'_\lambda(u) = 0$ has a nontrivial solution for almost every $\lambda \in \Lambda$. This is equivalent to (9.10) having a nontrivial solution for almost every $\beta \in (K^{-1}, \sigma^{-1})$. Since K was arbitrary, the result follows. □

To prove Theorem 9.6, it suffices to take $\tilde{\lambda} = \lambda_0$ and show that hypothesis **(D)** implies hypothesis $(\mathbf{D}'')$. To see this, we note that

$$\int_\Omega F(x, \pm R\varphi_0)\, dx/R^2 = \int_\Omega \frac{F(x, \pm R\varphi_0)}{R^2\varphi_0^2}\varphi_0^2 dx \to \infty$$

by hypothesis **(D)** and the fact that $\varphi_0(x) \neq 0$ a.e.

To prove Corollary 9.8, we let ε be any positive number. By hypothesis $(\mathbf{C}''')$, there is a $\delta > 0$ such that

$$F(x, t)/t^2 \leq \varepsilon, \quad |t| \leq \delta, \quad x \in \Omega.$$

By Theorem 9.7, (9.10) has a nontrivial solution for a.e. $\beta \in (0, \lambda_0/\varepsilon)$. Since ε was arbitrary, the result follows.

We now give the proof of Theorem 9.4.

Proof. By Theorem 2.4, for each arbitrary $K > 1$, and a.e. $\lambda \in (1, K)$, there exists u_λ such that $G'_\lambda(u_\lambda) = 0$, $G_\lambda(u_\lambda) = a(\lambda) \geq (\lambda - 1)\rho^2$. Choose $\lambda_n \to 1$, $\lambda_n > 1$. Then there exists u_n such that

$$G'_{\lambda_n}(u_n) = 0, \quad G_{\lambda_n}(u_n) = a(\lambda_n) \geq a(1) \geq b_0(1).$$

By Theorem 3.4, we may assume that $b_0(1) \geq \varepsilon > 0$. Therefore,

$$\int_\Omega \frac{2F(x, u_n)}{\|u_n\|_D^2} dx \leq c.$$

Now we prove that $\{u_n\}$ is bounded. If $\|u_n\|_D \to \infty$, let $w_n = u_n/\|u_n\|_D$; then $w_n \to w$ weakly in D, strongly in $L^2(\Omega)$, and a.e. in Ω.

We now have two cases:

Case 1: $w \neq 0$ in D. We get a contradiction as follows:

$$\begin{aligned} c &\geq \int_\Omega \frac{2F(x, u_n)}{\|u_n\|_D^2} dx = \int_\Omega \frac{2F(x, u_n)}{u_n^2} |w_n|^2 dx \\ &\geq \int_{w \neq 0} \frac{2F(x, u_n)}{u_n^2} |w_n|^2 dx - \int_{w=0} W_1(x)\, dx \to \infty. \end{aligned}$$

Case 2: $w = 0$ in D. We define $t_n \in [0, 1]$ by

$$G_{\lambda_n}(t_n u_n) = \max_{t \in [0,1]} G_{\lambda_n}(t u_n).$$

For any $c > 0$ and $\bar{w}_n = c w_n$, we have

$$\int_\Omega F(x, \bar{w}_n)\, dx \to 0$$

(cf., e.g., [122, p. 64]). Thus,

$$G_{\lambda_n}(t_n u_n) \geq G_{\lambda_n}(c w_n) = c^2 \lambda_n - 2\int_\Omega F(x, \bar{w}_n)\, dx \geq c^2/2$$

for n large enough. That is, $\lim_{n \to \infty} G_{\lambda_n}(t_n u_n) = \infty$ and $(G'_{\lambda_n}(t_n u_n), u_n) = 0$. Therefore,

$$\begin{aligned} G_{\lambda_n}(t_n u_n) &= \int_\Omega \Big(f(x, t_n u_n) t_n u_n - 2F(x, t_n u_n) \Big)\, dx \\ &= \int_\Omega H(x, t_n u_n)\, dx \to \infty. \end{aligned}$$

By hypothesis **(E′),**

$$G_{\lambda_n}(u_n) = \int_\Omega H(x, u_n)\, dx \geq \int_\Omega H(x, t_n u_n)\, dx \to \infty.$$

But

$$\begin{aligned} G_{\lambda_n}(u_n) = a(\lambda_n) &\le \sup_{s\in[0,1]\,,u\in A} G_{\lambda_n}((1-s)u) \\ &\le \sup_{s\in[0,1]\,,u\in A} G_K((1-s)u) \\ &< c, \end{aligned}$$

a contradiction. Thus, $\|u_n\|_D \le C$. It now follows that

$$G'(u_n) \to 0, \quad G(u_n) \to a(1) \ge b_0(1).$$

We can now apply Theorem 3.4.1 in [120, p. 64] to obtain the desired solution. □

9.6 The monotonicity trick

We now give the proof of Theorem 9.15.

Proof. First, we prove conclusion (1) with the first alternative $(\mathbf{H_1})$.

Evidently, $a(\lambda) \ge b_0(\lambda)$ since A links B. By $(\mathbf{H_1})$, the map $\lambda \mapsto a(\lambda)$ is non-decreasing. Hence, $a'(\lambda) := da(\lambda)/d\lambda$ exists for almost every $\lambda \in \Lambda$. From this point on, we consider those λ where $a'(\lambda)$ exists. For fixed $\lambda \in \Lambda$, let $\lambda_n \in (\lambda, 2\lambda) \cap \Lambda$, $\lambda_n \to \lambda$ as $n \to \infty$. Then there exists $\bar{n}(\lambda)$ such that

$$a'(\lambda) - 1 \le \frac{a(\lambda_n) - a(\lambda)}{\lambda_n - \lambda} \le a'(\lambda) + 1 \quad \text{for} n \ge \bar{n}(\lambda). \tag{9.31}$$

Next, we note that there exist $\Gamma_n \in \Phi$, $k_0 := k_0(\lambda) > 0$ such that

$$\|\Gamma_n(s)u\| \le k_0 \quad \text{whenever} \quad G_\lambda(\Gamma_n(s)u) \ge a(\lambda) - (\lambda_n - \lambda). \tag{9.32}$$

In fact, by the definition of $a(\lambda_n)$, there exists $\Gamma_n \in \Phi$ such that

$$\sup_{s\in[0,1],u\in A} G_\lambda(\Gamma_n(s)u) \le \sup_{s\in[0,1],u\in A} G_{\lambda_n}(\Gamma_n(s)u) \le a(\lambda_n) + (\lambda_n - \lambda). \tag{9.33}$$

If $G_\lambda(\Gamma_n(s)u) \ge a(\lambda) - (\lambda_n - \lambda)$ for some $u \in A, s \in [0, 1]$, then, by (9.31) and (9.33), we have that

$$\begin{aligned} I(\Gamma_n(s)u) &= \frac{G_{\lambda_n}(\Gamma_n(s)u) - G_\lambda(\Gamma_n(s)u)}{\lambda_n - \lambda} \\ &\le \frac{a(\lambda_n) + (\lambda_n - \lambda) - a(\lambda) + (\lambda_n - \lambda)}{\lambda_n - \lambda} \\ &\le a'(\lambda) + 3, \end{aligned} \tag{9.34}$$

and it follows that

$$\begin{aligned} J(\Gamma_n(s)u) &= \lambda_n I(\Gamma_n(s)u) - G_{\lambda_n}(\Gamma_n(s)u) \\ &\le \lambda_n(a'(\lambda) + 3) - G_\lambda(\Gamma_n(s)u) \\ &\le \lambda_n(a'(\lambda) + 3) - a(\lambda) + (\lambda_n - \lambda) \\ &\le 2\lambda(a'(\lambda) + 3) - a(\lambda) + \lambda. \end{aligned} \tag{9.35}$$

On the other hand, by $(\mathbf{H_1})$, (9.31), and (9.33),

$$\begin{aligned} J(\Gamma_n(s)u) &= \lambda_n I(\Gamma_n(s)u) - G_{\lambda_n}(\Gamma_n(s)u) \\ &\geq -G_{\lambda_n}(\Gamma_n(s)u) \\ &\geq -(a(\lambda_n) + (\lambda_n - \lambda)) \\ &\geq -(a(\lambda) + (\lambda_n - \lambda)(a'(\lambda) + 2)) \\ &\geq -a(\lambda) - \lambda|a'(\lambda) + 2|. \end{aligned} \tag{9.36}$$

Combining (9.34)–(9.37) and $(\mathbf{H_1})$, we see that there exists $k_0(\lambda) := k_0$ (depending only on λ) such that (9.32) holds.

First, we consider the case of $a(\lambda) > b_0(\lambda)$. For each n, we define

$$Q_n(\lambda) := \{u \in E : \|u\| \leq k_0 + 1,\ |G_\lambda(u) - a(\lambda)| \leq \lambda_n - \lambda\}. \tag{9.37}$$

We claim that $Q_n(\lambda) \neq \phi$ and that

$$\inf\{\|G'_\lambda(u)\| : u \in Q_n(\lambda)\} \to 0 \tag{9.38}$$

as $n \to \infty$.

By the definition of $a(\lambda)$, there exists $(s_0, u_0) \in [0, 1] \times A$ such that

$$G_\lambda(\Gamma_n(s_0)u_0) \geq a(\lambda) - (\lambda_n - \lambda).$$

Then, by (9.32), $\|\Gamma_n(s_0)u_0\| \leq k_0(\lambda)$. It follows that $\Gamma_n(s_0)u_0 \in Q_n(\lambda)$. Therefore, $Q_n(\lambda) \neq \phi$.

Next, we want to prove (9.38). If this were not so, there would exist a positive $\varepsilon < [a(\lambda) - b_0(\lambda)]/3$ such that $\|G'_\lambda(u)\| \geq 3\varepsilon$ for all $u \in Q_n(\lambda)$. We take n so large that $(a'(\lambda)+2)(\lambda_n - \lambda) \leq \varepsilon$, $\lambda_n - \lambda \leq \varepsilon$. By Lemma 2.10.1 of [122], we may construct a locally Lipschitz continuous map Y_λ of $\hat{E}$ such that

1. $\|Y_\lambda(u)\| \leq 1, \quad \forall u \in \hat{E}$,
2. $(G'_\lambda(u), Y_\lambda(u)) \geq 2\varepsilon, \quad \forall u \in Q_n(\lambda)$,
3. $(G'_\lambda(u), Y_\lambda(u)) \geq 0, \quad \forall u \in \hat{E}$.

Consider the initial boundary-value problem:

$$\frac{d\sigma(t)u}{dt} = -Y_\lambda(\sigma(t)u), \quad \sigma(0, u) = u.$$

By Theorem 4.5, there exists a unique continuous solution $\sigma(t)u$ such that $G_\lambda(\sigma(t)u)$ is nonincreasing in t. Define

$$\tilde{\Gamma}(s)u := \begin{cases} \sigma(2s)u, & 0 \leq s \leq 1/2, \\ \sigma(1)\Gamma_n(2s-1)u, & 1/2 \leq s \leq 1. \end{cases}$$

Then it is easy to check that $\tilde{\Gamma} \in \Phi$. We want to prove that

$$G_\lambda(\tilde{\Gamma}(s)u) \leq a(\lambda) - (\lambda_n - \lambda), \quad \forall (s, u) \in [0, 1] \times A, \tag{9.39}$$

which provides the desired contradiction. Choose any $u \in A$. If $0 \leq s \leq 1/2$, by (**H$_3$**), we get

$$
\begin{aligned}
G_\lambda(\tilde{\Gamma}(s)u) &= G_\lambda(\sigma(2s)u) \\
&\leq G_\lambda(u) \\
&\leq a_0(\lambda) \\
&\leq b_0(\lambda) \\
&< a(\lambda) - (\lambda_n - \lambda).
\end{aligned} \tag{9.40}
$$

If $1/2 \leq s \leq 1$, then $\tilde{\Gamma}(s)u = \sigma(1)\Gamma_n(2s-1)u$. If $G_\lambda(\Gamma_n(2s-1)u) < a(\lambda) - (\lambda_n - \lambda)$ for $s \in [1/2, 1]$, then

$$
\begin{aligned}
G_\lambda(\tilde{\Gamma}(s)u) &= G_\lambda(\sigma(1)\Gamma_n(2s-1)u) \\
&\leq G_\lambda(\sigma(0)\Gamma_n(2s-1)u) \\
&< a(\lambda) - (\lambda_n - \lambda).
\end{aligned} \tag{9.41}
$$

If there exists an $s_0 \in [1/2, 1]$ such that $G_\lambda(\Gamma_n(2s_0 - 1)u) \geq a(\lambda) - (\lambda_n - \lambda)$, then, by (9.32), $\|\Gamma_n(2s_0 - 1)u\| \leq k_0$. Since

$$\|\sigma(t)u - u\| \leq t,$$

it follows that

$$\|\sigma(t)\Gamma_n(2s_0 - 1)u\| \leq \|\Gamma_n(2s_0 - 1)u\| + t \leq k_0 + 1 \quad \text{for all } t \in [0, 1]. \tag{9.42}$$

If there is a $t_0 \in [0, 1]$ such that

$$G_\lambda(\sigma(t_0)\Gamma_n(2s_0 - 1)u) < a(\lambda) - (\lambda_n - \lambda) \quad \text{for some } t_0 \in [0, 1],$$

we obtain

$$G_\lambda(\tilde{\Gamma}(s_0)u) = G_\lambda(\sigma(1)\Gamma_n(2s_0 - 1)u) < a(\lambda) - (\lambda_n - \lambda). \tag{9.43}$$

If there is no such $t_0 \in [0, 1]$, then in view of (9.32), we see that

$$
\begin{aligned}
a(\lambda) - (\lambda_n - \lambda) &\leq G_\lambda(\sigma(1)\Gamma_n(2s_0 - 1)u) \\
&\leq G_\lambda(\sigma(t)\Gamma_n(2s_0 - 1)u) \\
&\leq G_\lambda(\Gamma_n(2s_0 - 1)u) \\
&\leq a(\lambda) - (\lambda_n - \lambda)
\end{aligned} \tag{9.44}
$$

for all $t \in [0, 1]$. Thus, (9.42) and (9.44) imply that $\sigma(t)\Gamma_n(2s_0 - 1)u \in Q_n(\lambda)$ for all $t \in [0, 1]$. Since $(G'_\lambda(u), Y_\lambda(u)) \geq 2\varepsilon$ on $Q_n(\lambda)$, we have

$$\begin{aligned}
&G_\lambda(\sigma(t)\Gamma_n(2s_0 - 1)u) - G_\lambda(\Gamma_n(2s_0 - 1)u) \\
&= \int_0^t \frac{dG_\lambda(\sigma(\theta)\Gamma_n(2s_0 - 1)u)}{d\theta} d\theta \\
&\leq \int_0^t -\Big(G'_\lambda(\sigma(\theta)\Gamma_n(2s_0 - 1)u), Y_\lambda(\sigma(\theta)\Gamma_n(2s_0 - 1)u)\Big) d\theta \\
&\leq -2t\varepsilon.
\end{aligned}$$

Therefore,

$$\begin{aligned}
G_\lambda(\tilde{\Gamma}(s_0)u) &= G_\lambda(\sigma(1)\Gamma_n(2s_0 - 1)u) \\
&\leq G_\lambda(\Gamma_n(2s_0 - 1)u) - 2\varepsilon \\
&\leq a(\lambda) - (\lambda_n - \lambda).
\end{aligned} \tag{9.45}$$

Combining (9.40), (9.41), (9.43), and (9.45), we get

$$G_\lambda(\tilde{\Gamma}(s)u) \leq a(\lambda) - (\lambda_n - \lambda), \quad \forall (s, u) \in [0, 1] \times A,$$

which contradicts the definition of $a(\lambda)$. This implies that (9.38) holds in the case of $a(\lambda) > b_0(\lambda)$. Clearly, (9.38) yields conclusion (1) of this theorem.

We prove that conclusion (1) of Theorem 9.15 is still true in case $a(\lambda) = b_0(\lambda)$. Since A is bounded, $d_A := \max\{\|u\| : u \in A\} < \infty$. For $\varepsilon > 0$, $T > 0$, we define

$$\begin{aligned}
\bar{Q}(\varepsilon, T, \lambda) :=& \{u \in E : \|u\| \leq k_0(\lambda) + 4 + d_A, \\
& |G_\lambda(u) - a(\lambda)| \leq 3\varepsilon,\ d(u, B) \leq 4T\}.
\end{aligned}$$

We claim that $\bar{Q}(\varepsilon, T, \lambda) \neq \phi$. By (9.33), we may choose n so large that

$$\sup_{s\in[0,1], u\in A} G_\lambda(\Gamma_n(s)u) \leq \sup_{s\in[0,1], u\in A} G_{\lambda_n}(\Gamma_n(s)u) \leq a(\lambda) + 3\varepsilon. \tag{9.46}$$

Since A links B [hm], there exists $(s_0, u_0) \in [0, 1] \times A$ such that $\Gamma_n(s_0)u_0 \in B$. Hence $\operatorname{dist}(\Gamma_n(s_0)u_0, B) = 0$ and

$$G_\lambda(\Gamma_n(s_0)u_0) \geq b_0(\lambda) = \inf_B G_\lambda = a(\lambda) > a(\lambda) - (\lambda_n - \lambda) \geq a(\lambda) - 3\varepsilon. \tag{9.47}$$

By (9.32), $\|\Gamma_n(s_0)u_0\| \leq k_0$. Hence, $\Gamma_n(s_0)u_0 \in \bar{Q}(\varepsilon, T, \lambda)$.

Next, we prove that

$$\inf\{\|G'_\lambda(u)\| : u \in \bar{Q}(\varepsilon, T, \lambda)\} = 0 \tag{9.48}$$

for ε, T sufficiently small. If not, there would exist $\delta > 0$, $\varepsilon_1 > 0$, and $T_1 \in (0, 1)$ such that

$$\|G'_\lambda(u)\| \geq 3\delta \quad \text{for } u \in \bar{Q}(\varepsilon_1, T_1, \lambda). \tag{9.49}$$

Define

$$\bar{Q}^*(\varepsilon_1, T_1, \lambda) := \{u \in E : \|u\| \le k_0 + 4 + d_A,\ \operatorname{dist}(u, B) \le 4T_1,$$
$$a(\lambda) - (\lambda_n - \lambda) \le G_\lambda(u) \le a(\lambda) + 3\varepsilon_1\}. \tag{9.50}$$

By (9.46) and (9.47), $\bar{Q}^*(\varepsilon_1, T_1, \lambda) \neq \phi$ and $\bar{Q}^*(\varepsilon_1, T_1, \lambda) \subset \bar{Q}(\varepsilon_1, T_1, \lambda)$. Let n be so large that $\lambda_n - \lambda \le \varepsilon_1$, $(a'(\lambda) + 2)(\lambda_n - \lambda) < \varepsilon_1$ and $\lambda_n - \lambda < \delta T_1$. Similarly, we may construct a locally Lipschitz continuous map $\bar{Y}_\lambda$ of $\hat{E}$ such that

1. $\|\bar{Y}_\lambda(u)\| \le 1, \quad \forall u \in \hat{E}$,
2. $(G'_\lambda(u), \bar{Y}_\lambda(u)) \ge 2\delta, \quad \forall u \in \bar{Q}(\varepsilon_1, T_1, \lambda)$,
3. $(G'_\lambda(u), \bar{Y}_\lambda(u)) \ge 0, \quad \forall u \in \hat{E}$.

Define

$$Q_1 := \{u \in E : \|u\| \le k_0 + 2 + d_A, |G_\lambda(u) - a(\lambda)| \le 2\varepsilon_1, \operatorname{dist}(u, B) \le 3T_1\}. \tag{9.51}$$

As in the proof of (9.38), $Q_1 \neq \phi$ and $Q_1 \subset \bar{Q}(\varepsilon_1, T_1, \lambda)$. Moreover, the distance between Q_1 and $E \backslash \bar{Q}(\varepsilon_1, T_1, \lambda)$ is positive. Thus, we may choose a Lipschitz continuous map γ from E into $[0, 1]$ that equals 1 on Q_1 and vanishes outside $\bar{Q}(\varepsilon_1, T_1, \lambda)$. Consider the following initial boundary-value problem:

$$\frac{d(\sigma_1(t)u)}{dt} = -\gamma(\sigma_1)\bar{Y}_\lambda(\sigma_1), \quad \sigma_1(0)u = u.$$

Let $\sigma_1(t)u$ be the unique continuous solution. Then we have that

$$\frac{dG_\lambda(\sigma_1(t)u)}{dt} \le -2\delta\gamma(\sigma_1(t)u) \le 0. \tag{9.52}$$

First, we note that

$$\sigma_1(t)u \notin B, \quad s \in [0, T_1],\ u \in A. \tag{9.53}$$

For $u \in A$, by (9.52), we have that

$$G_\lambda(\sigma_1(t)u) \le G_\lambda(u) \le a_0(\lambda) \le b_0(\lambda) = a(\lambda), \quad \forall t \in [0, T_1], \tag{9.54}$$

and

$$\begin{aligned} G_\lambda(\sigma_1(t)u) &= G_\lambda(u) + \int_0^t \frac{dG_\lambda(\sigma_1(\theta)u)}{d\theta} d\theta \\ &\le G_\lambda(u) - \int_0^t 2\delta\gamma(\sigma_1(\theta)u) d\theta \end{aligned} \tag{9.55}$$

for all $t \in [0, T_1]$.

If (9.53) were not true, then there would be a $t_0 \in [0, T_1]$ such that $\sigma_1(t_0)u \in B$. Then $G_\lambda(\sigma_1(t_0)u) \geq a(\lambda) = b_0(\lambda) = \inf_B G_\lambda$. By (9.52)–(9.56), we see that

$$\int_0^{t_0} 2\delta\gamma(\sigma_1(\theta)u)d\theta = 0.$$

Hence, $\gamma(\sigma_1(\theta)u) = 0$ for $\theta \in [0, t_0]$; i.e., $\sigma_1(\theta)u \notin Q_1\ \forall\theta \in [0, t_0]$. Therefore, one of the following three cases occurs:

$$\|\sigma_1(\theta)u\| > k_0 + 2 + d_A; \tag{9.56}$$

or

$$|G_\lambda(\sigma_1(\theta)u) - a(\lambda)| > 2\varepsilon_1; \tag{9.57}$$

or

$$\mathrm{dist}(\sigma_1(\theta)u, B) > 3T_1. \tag{9.58}$$

Inequality (9.56) cannot hold, since

$$\|\sigma_1(\theta, u) - \sigma_1(\theta')u\| \leq |\theta - \theta'| \tag{9.59}$$

and

$$\|\sigma_1(\theta)u\| \leq \|\sigma_1(0)u\| + T_1 \leq d_A + 1, \quad \forall\theta \in [0, t_0].$$

If (9.57) holds, then $G_\lambda(\sigma_1(\theta)u) < a(\lambda) - 2\varepsilon_1$. Hence, $\sigma_1(\theta)u \notin B$. Evidently, (9.58) implies that $\sigma_1(\theta)u \notin B$. Therefore, (9.53) is true.

Next, we note that

$$\sigma_1(T_1)\Gamma_n(2s-1)u \notin B, \quad u \in A,\ s \in [1/2, 1]. \tag{9.60}$$

For any fixed $u \in A$ and $s \in [1/2, 1]$, we divide the proof into two cases.

Case 1: If $\sigma_1(\theta)\Gamma_n(2s-1)u \in Q_1$ for all $\theta \in [0, T_1]$, by (9.52) and (9.32), we have that

$$\begin{aligned}
&G_\lambda(\sigma_1(T_1)\Gamma_n(2s-1)u)\\
&= G_\lambda(\Gamma_n(2s-1)u) + \int_0^{T_1} \frac{dG_\lambda(\sigma_1(\theta)\Gamma_n(2s-1)u)}{d\theta}d\theta\\
&\leq G_\lambda(\Gamma_n(2s-1)u) - \int_0^{T_1} 2\delta\gamma(\sigma_1(\theta)\Gamma_n(2s-1)u)d\theta\\
&= G_\lambda(\Gamma_n(2s-1)u) - 2\delta T_1\\
&\leq a(\lambda) - 2\delta T_1 + (a'(\lambda) + 2)(\lambda_n - \lambda),
\end{aligned}$$

which implies that $\sigma_1(T_1)\Gamma_n(2s-1)u \notin B$ since $a(\lambda) = b_0(\lambda)$.

Case 2: If there exists $t_0 \in [0, T_1]$ such that $\sigma_1(t_0)\Gamma_n(2s-1)u \notin Q_1$, then one of the following alternatives holds:

either

$$\|\sigma_1(t_0)\Gamma_n(2s-1)u\| > k_0 + 2 + d_A; \tag{9.61}$$

or

$$|G_\lambda(\sigma_1(t_0)\Gamma_n(2s-1)u) - a(\lambda)| > 2\varepsilon_1; \tag{9.62}$$

or

$$\operatorname{dist}(\sigma_1(t_0)\Gamma_n(2s-1)u, B) > 3T_1. \tag{9.63}$$

Assume that (9.61) holds. If $\sigma_1(T_1)\Gamma_n(2s-1)u \in B$, then

$$b_0(\lambda) = a(\lambda) \le G_\lambda(\sigma_1(T_1)\Gamma_n(2s-1)u) \le G_\lambda(\Gamma_n(2s-1)u).$$

By (9.32), $\|\Gamma_n(2s-1)u\| \le k_0$. Furthermore, since

$$\|\sigma_1(t_0)\Gamma_n(2s-1)u - \sigma_1(0)\Gamma_n(2s-1)u\| \le t_0,$$

it follows that

$$\|\sigma_1(t_0)\Gamma_n(2s-1)u\| \le k_0 + t_0 \le k_0 + 1,$$

which contradicts (9.61). Hence, $\sigma_1(T_1)\Gamma_n(2s-1)u \notin B$.

Assume that (9.62) holds. Note, by (9.32), that

$$\begin{aligned} G_\lambda(\sigma_1(t_0)\Gamma_n(2s-1)u) &\le G_\lambda(\sigma_1(0)\Gamma_n(2s-1)u) \\ &= G_\lambda(\Gamma_n(2s-1)u) \\ &\le a(\lambda) + \varepsilon_1. \end{aligned}$$

Therefore, (9.62) implies that

$$G_\lambda(\sigma_1(T_1)\Gamma_n(2s-1)u) \le G_\lambda(\sigma_1(t_0)\Gamma_n(2s-1)u) \le a(\lambda) - 2\varepsilon_1.$$

It follows that $\sigma_1(T_1)\Gamma_n(2s-1)u \notin B$ since $a(\lambda) = b_0(\lambda)$.

Assume (9.63) holds. Note that $\|\sigma_1(t)u - \sigma_1(t')u\| \le |t-t'|$. It therefore follows that

$$\|\sigma_1(t)\Gamma_n(2s-1)u - w\| \ge \|\sigma_1(t_0)\Gamma_n(2s-1)u - w\| - |t-t_0|$$

for all $w \in B, t \in [0, T_1]$. Hence, $\operatorname{dist}(\sigma_1(t)\Gamma_n(2s-1)u, B) \ge 2T_1$ for all $t \in [0, T_1]$. In particular,

$$\sigma_1(T_1)\Gamma_n(2s-1)u \notin B.$$

Hence, (9.60) holds in this case as well.

Define

$$\Gamma_1^*(s)u := \begin{cases} \sigma_1(2sT_1)u, & 0 \le s \le 1/2, \\ \sigma_1(T_1)\Gamma_n(2s-1)u, & 1/2 \le s \le 1. \end{cases}$$

Then $\Gamma_1^* \in \Phi$. Moreover, by (9.53) and (9.60), $\Gamma_1^*(s)A \cap B = \phi$ for all $s \in [0, 1]$. This gives the contradiction that completes the proof.

Now, we consider conclusion (1) with the second alternative $(\mathbf{H}_2)$.

For this case, the map $\lambda \mapsto a(\lambda)$ is nonincreasing and $a'(\lambda) = da(\lambda)/d\lambda$ exists for almost all $\lambda > 0$. Therefore, we consider those λ where $a'(\lambda)$ exists. We choose $\lambda_n \in (0, \lambda) \cap \Lambda$ and $\lambda_n \to \lambda$ as $n \to \infty$. Then (9.31) is still true for n large enough. We also prove that there exists a $\Gamma_n \in \Phi$, $k_0 = k_0(\lambda) > 0$ such that

$$\|\Gamma_n(s)u\| \le k_0 \ \text{ if } \ G_\lambda(\Gamma_n(s, u)) \ge a(\lambda) - (\lambda - \lambda_n). \tag{9.64}$$

In fact, by the definition of $a(\lambda_n)$, there exists a $\Gamma_n \in \Phi$ such that

$$\begin{aligned} \sup_{s\in[0,1],u\in A} G_\lambda(\Gamma_n(s)u) &\le \sup_{s\in[0,1],u\in A} G_{\lambda_n}(\Gamma_n(s)u) \\ &\le a(\lambda_n) + (\lambda - \lambda_n) \\ &\le a(\lambda) + (a'(\lambda) - 2)(\lambda_n - \lambda). \end{aligned} \tag{9.65}$$

If $G_\lambda(\Gamma_n(s)u) \ge a(\lambda) - (\lambda - \lambda_n)$, then by (9.31), (9.64), and (9.65),

$$\begin{aligned} I(\Gamma_n(s, u)) &= \frac{G_{\lambda_n}(\Gamma_n(s)u) - G_\lambda(\Gamma_n(s)u)}{\lambda_n - \lambda} \\ &\ge \frac{a(\lambda_n) - a(\lambda) + 2(\lambda - \lambda_n)}{\lambda_n - \lambda} \\ &\ge a'(\lambda) - 3. \end{aligned} \tag{9.66}$$

On the other hand, by $(\mathbf{H_2})$, (9.31), (9.65), (9.66),

$$\begin{aligned} J(\Gamma_n(s)u) &= \lambda_n I(\Gamma_n(s)u) - G_{\lambda_n}(\Gamma_n(s)u) \\ &\ge -\lambda|a'(\lambda) - 3| - \lambda - a(\lambda_n) \\ &\ge -\lambda|a'(\lambda) - 3| - \lambda - \lambda|a'(\lambda) - 1| - a(\lambda) \end{aligned} \tag{9.67}$$

and

$$J(\Gamma_n(s)u) \le -G_{\lambda_n}(\Gamma_n(s)u) \le -G_\lambda(\Gamma_n(s)u) \le -a(\lambda) + \lambda. \tag{9.68}$$

Hence, (9.66)–(9.68) imply that $\|\Gamma_n(s)u\| \le k_0 = k_0(\lambda)$, a constant depending only on λ. The rest is similar to the proof under the assumption $(\mathbf{H}_1)$; we omit the details.

Finally, conclusion (2) can be proved immediately by interchanging A and B, replacing G_λ by $-G_\lambda$, and using conclusion (1). □

9.7 Notes and remarks

Problem (9.6) has been studied by many people. The vast majority of results obtained concern sublinear problems. Much less has been proved for the superlinear case. In [7] the basic assumption was

$$0 < \mu F(x, t) \le tf(x, t), \quad |t| \ge r, \tag{9.69}$$

for some $\mu > 2$ and $r \geq 0$. This is a very convenient hypothesis since it readily achieves mountain pass geometry as well as satisfaction of the Palais–Smale condition. However, it is a severe restriction; it strictly controls the growth of $f(x,t)$ as $|t| \to \infty$. Almost every author discussing superlinear problems has made this assumption. We have been able to weaken assumption (9.69) considerably, but not to our complete satisfaction. We assume either that

$$\mu F(x,t) - tf(x,t) \leq C(t^2 + 1), \quad |t| \geq r,$$

for some $\mu > 2$ and $r \geq 0$ or that (9.7) is convex in t. These allow much more freedom for the function $f(x,t)$. But they do not allow as much freedom as we would like. We were able to weaken it much further and assume only hypothesis **(D)**. However, we paid a heavy price for this generalization: We were only able to solve (9.10) for almost all positive values of β.

The method (called the monotonicity trick), which allowed us to solve (9.10) for almost all values of β in some interval, was first introduced by Struwe [148] for minimization problems. It was applied by Jeanjean [76] and others for various types of problems.

The material of this chapter comes from [141] and [139]. See also [49].

Chapter 10

Weak Linking

10.1 Introduction

As we noted, a subset A of a Banach space E links a subset B of E if, for every $G \in C^1(E, \mathbb{R})$ satisfying

(10.1) $$a_0 := \sup_A G \le b_0 := \inf_B G,$$

there are a sequence $\{u_k\} \subset E$ and a constant c such that

(10.2) $$b_0 \le c < \infty$$

and

(10.3) $$G(u_k) \to c, \quad G'(u_k) \to 0.$$

We also saw that there are several criteria that imply that a set A links a set B. However, all of them require that at least one of the sets A, B be contained in a finite-dimensional manifold. It is not clear if it is possible for A to link B if neither is contained in such a manifold. For instance, if $E = M \oplus N$, where M, N are closed, infinite-dimensional subspaces of E and B_R is the ball centered at the origin of radius R in E, it is unknown if the set $A = M \cap \partial B_R$ links $B = N$. (If either M or N is finite-dimensional, then A does link B; cf. Example 2 of Section 3.4.) Very little is known for the infinite-dimensional case. Unfortunately, this situation arises in some important applications, including Hamiltonian systems, the wave equation, and elliptic systems, to name a few.

The purpose of the present chapter is to study linking when both M and N are infinite-dimensional and G' has some additional continuity property. The property we have chosen is that of weak-to-weak continuity:

Definition 10.1. *Let E be a Banach space. We shall call a functional $G \in C^1(E, \mathbb{R})$* **weak-to-weak continuously differentiable** *if, for each sequence*

(10.4) $$u_k \to u \text{ weakly in } E,$$

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there exists a renamed subsequence such that

$$G'(u_k) \to G'(u) \text{ weakly.} \tag{10.5}$$

We restrict our attention to Hilbert space and prove the following.

Theorem 10.2. *Let E be a separable Hilbert space, and let G be a weak-to-weak continuously differentiable functional on E. Let N be a closed subspace of E, and let Q be a bounded, convex, open subset of N containing the point p. Let F be a continuous map of E onto N such that*

(a) $F|_Q = I$,

and

(b) for each finite-dimensional subspace S of E containing p such that $FS \neq \{0\}$, there is a finite-dimensional subspace $S_0 \neq \{0\}$ of N containing p such that

$$v \in \bar{Q} \cap S_0, \quad w \in S \Longrightarrow F(v + w) \in S_0. \tag{10.6}$$

(The restriction $FS \neq \{0\}$ is made in case $p = 0$.) Set $A = \partial Q,\ B = F^{-1}(p)$. If

$$a_1 = \sup_{\bar{Q}} G < \infty \tag{10.7}$$

and (10.1) holds, then there is a sequence $\{u_k\} \subset E$ such that (10.2), (10.3) hold and $c \leq a_1$.

If F does not satisfy (a), but the restriction F_0 of F to $\bar{Q}$ is a homeomorphism of $\bar{Q}$ onto the closure of a convex, open subset $\Omega \subset N$, then this can replace (a). Moreover, most, if not all, sets A, B known to link when one of the subspaces M, N is finite-dimensional will link now as well.

This leads to the following.

Definition 10.3. *A subset A of a Banach space E links a subset B* **weakly** *if, for every $G \in C^1(E, \mathbb{R})$ that is weak-to-weak continuously differentiable and satisfies (10.1), there are a sequence $\{u_k\} \subset E$ and a constant c such that (10.2) and (10.3) hold.*

Thus we have

Corollary 10.4. *Under the hypotheses of Theorem 10.2, the set $A = \partial Q$ links the set $B = F^{-1}(p)$ weakly.*

Theorem 10.2 will be proved in the next section.

10.2 Another norm

In this section we prove Theorem 10.2 by introducing a different norm that is equivalent to the weak topology on bounded sets.

Proof. Assume that there is no such sequence. Then there is a positive number $\delta > 0$ such that

$$\|G'(u)\| \geq 2\delta \tag{10.8}$$

whenever u belongs to the set

$$\hat{E} = \{u \in E : b_0 - 2\delta \leq G(u) \leq a_1 + 2\delta\}. \tag{10.9}$$

Since E is separable, we can norm it with a norm $|u|_w$ satisfying

$$|u|_w \leq \|u\|, \quad u \in E, \tag{10.10}$$

and such that the topology induced by this norm is equivalent to the weak topology of E on bounded subsets of E.

This can be done as follows. Let $\{e_k\}$ be an orthonormal basis for E. We then set

$$|u|_w^2 = \sum_{k=1}^{\infty} \frac{|(u, e_k)|^2}{k^2}.$$

We denote E equipped with this norm by $\tilde{E}$. Let

$$E' = \{u \in E : G'(u) \neq 0\}.$$

For $u \in E'$, let $h(u) = G'(u)/\|G'(u)\|$. Then, by (10.8),

$$(G'(u), h(u)) \geq 2\delta, \quad u \in \hat{E}. \tag{10.11}$$

Let

$$\begin{aligned} T &= (a_1 - b_0 + 4\delta)/\delta, \\ B_R &= \{u \in E : \|u\| < R\}, \\ R &= \sup_Q \|u\| + T, \\ \hat{B} &= \bar{B}_R \cap \hat{E}. \end{aligned} \tag{10.12}$$

For each $u \in \hat{B}$, there is an $\tilde{E}$ neighborhood $W(u)$ of u such that

$$(G'(v), h(u)) > \delta, \quad v \in W(u) \cap \hat{B}. \tag{10.13}$$

Otherwise, there would be a sequence $\{v_k\} \subset \hat{B}$ such that

$$|v_k - u|_w \to 0 \quad \text{and} \quad (G'(v_k), h(u)) \leq \delta. \tag{10.14}$$

Since $\hat{B}$ is bounded in E, $v_k \to u$ weakly in E and (10.5) implies that

$$(G'(v_k), h(u)) \to (G'(u), h(u)) \leq \delta \tag{10.15}$$

in view of (10.14). This contradicts (10.11). Let $\tilde{B}$ be the set $\hat{B}$ with the inherited topology of $\tilde{E}$. It is a metric space, and $W(u) \cap \tilde{B}$ is an open set in this space. Thus, $\{W(u) \cap \tilde{B}\}$, $u \in \tilde{B}$, is an open covering of the paracompact space $\tilde{B}$ (cf., e.g., [112]). Consequently, there is a locally finite refinement $\{W_\tau\}$ of this cover. For each τ, there is an element u_τ such that $W_\tau \subset W(u_\tau)$. Let $\{\psi_\tau\}$ be a partition of unity subordinate to this covering. Each ψ_τ is locally Lipschitz continuous with respect to the norm $|u|_w$ and consequently with respect to the norm of E. Let

$$Y(u) = \sum \psi_\tau(u) h(u_\tau), \quad u \in \tilde{B}. \tag{10.16}$$

Then $Y(u)$ is locally Lipschitz continuous with respect to both norms. Moreover,

$$\|Y(u)\| \leq \sum \psi_\tau(u) \|h(u_\tau)\| \leq 1 \tag{10.17}$$

and

$$(G'(u), Y(u)) = \sum \psi_\tau(u)(G'(u), h(u_\tau)) \geq \delta, \quad u \in \hat{B}. \tag{10.18}$$

For $u \in \bar{Q} \cap \hat{E}$, let $\sigma(t)u$ be the solution of

$$\sigma'(t) = -Y(\sigma(t)), \quad t \geq 0, \ \sigma(0) = u. \tag{10.19}$$

Note that $\sigma(t)u$ will exist as long as $\sigma(t)u$ is in $\hat{B}$. Moreover, it is continuous in (u, t) with respect to both topologies.

Next we note that if $u \in \bar{Q} \cap \hat{E}$, we cannot have $\sigma(t)u \in \hat{B}$ and $G(\sigma(t)u) > b_0 - \delta$ for $0 \leq t \leq T$: for by (10.18), (10.19),

$$dG(\sigma(t)u)/dt = (G'(\sigma), \sigma') = -(G'(\sigma), Y(\sigma)) \leq -\delta \tag{10.20}$$

as long as $\sigma(t)u \in \hat{B}$. Hence, if $\sigma(t)u \in \hat{B}$ for $0 \leq t \leq T$, we would have

$$G(\sigma(T)u) - G(u) \leq -\delta T = -(a_1 - b_0 + 4\delta). \tag{10.21}$$

Thus, we would have $G(\sigma(T)u) < b_0 - 4\delta$. On the other hand, if $\sigma(s)u$ exists for $0 \leq s < T$, then $\sigma(t)u \in \hat{B}$. To see this, note that

$$u - \sigma(t)u = z_t(u) := \int_0^t Y(\sigma(s)u)ds. \tag{10.22}$$

By (10.17),

$$\|z_t(u)\| \leq t. \tag{10.23}$$

Consequently,

$$\|\sigma(t)u\| \leq \|u\| + t < R. \tag{10.24}$$

Thus, $\sigma(t)u \in \hat{B}$. We can now conclude that for each $u \in \bar{Q} \cap \hat{E}$, there is a $t \geq 0$ such that $\sigma(s)u$ exists for $0 \leq s \leq t$ and $G(\sigma(t)u) \leq b_0 - \delta$. Let

$$T_u := \inf\{t \geq 0 : G(\sigma(t)u) \leq b_0 - \delta\}, \quad u \in \bar{Q} \cap \hat{E}. \tag{10.25}$$

Then $\sigma(t)u$ exists for $0 \le t \le T_u$ and $T_u < T$. Moreover, T_u is continuous in u. Define

$$\hat{\sigma}(t)u = \begin{cases} \sigma(t)u, & 0 \le t \le T_u, \\ \sigma(T_u)u, & T_u \le t \le T, \end{cases}$$

for $u \in \bar{Q} \cap \hat{E}$. For $u \in \bar{Q} \setminus \hat{E}$, define $\hat{\sigma}(t)u = u,\ \ 0 \le t \le T$. Then $\hat{\sigma}(t)u$ is continuous in (u, t), and

$$G(\hat{\sigma}(T)u) \le b_0 - \delta, \quad u \in \bar{Q}. \tag{10.26}$$

Let

$$\varphi(v, t) = F\hat{\sigma}(t)v, \quad v \in \bar{Q},\ 0 \le t \le T. \tag{10.27}$$

Then φ is a continuous map of $\bar{Q} \times [0, T]$ to N. Let

$$K = \{(u, t) : u = \hat{\sigma}(t)v,\ v \in \bar{Q},\ t \in [0, T]\}.$$

Then K is a compact subset of $\tilde{E} \times \mathbb{R}$. To see this, let (u_k, t_k) be any sequence in K. Then $u_k = \sigma(t_k)v_k$, where $v_k \in \bar{Q}$. Since Q is bounded, there is a subsequence such that $v_k \to v_0$ weakly in E and $t_k \to t_0$ in $[0, T]$. Since $\bar{Q}$ is convex and bounded, v_0 is in $\bar{Q}$ and $|v_k - v_0|_w \to 0$. Since $\hat{\sigma}(t)$ is continuous in $\tilde{E} \times \mathbb{R}$, we have

$$u_k = \hat{\sigma}(t_k)v_k \rightharpoonup \hat{\sigma}(t_0)v_0 = u_0 \in K.$$

Each $u_0 \in \hat{B}$ has a neighborhood $W(u_0)$ in $\tilde{E}$ and a finite-dimensional subspace $S(u_0)$ such that $Y(u) \subset S(u_0)$ for $u \in W(u_0) \cap \hat{B}$. Since $\hat{\sigma}(t)u$ is continuous in (u, t), for each $(u_0, t_0) \in K$ there are a neighborhood $W(u_0, t_0) \subset \tilde{E} \times \mathbb{R}$ and a finite-dimensional subspace $S(u_0, t_0) \subset E$ such that $\hat{z}_t(u) \subset S(u_0, t_0)$ for $(u, t) \in W(u_0, t_0)$, where

$$\hat{z}_t(u) := u - \hat{\sigma}(t)u = \begin{cases} \int_0^t Y(\hat{\sigma}(s)u)ds, & u \in \hat{E}, \\ 0, & u \notin \hat{E}. \end{cases} \tag{10.28}$$

Since K is compact, there are a finite number of points $(u_j, t_j) \in K$ such that $K \subset W = \cup W(u_j, t_j)$. Let S be a finite-dimensional subspace of E containing p and all the $S(u_j, t_j)$ and such that $FS \neq \{0\}$. Then, for each $v \in \bar{Q}$, we have $\hat{z}_t(v) \in S$. Then, by assumption (b) of Theorem 10.2, there is a finite dimensional-subspace $S_0 \neq \{0\}$ of N containing p such that $F(v - \hat{z}_t(v)) \in S_0$ for all $v \in \bar{Q} \cap S_0$. We note that $\varphi(u, t)$ maps $\bar{Q} \cap S_0 \times [0, T]$ into S_0. For t in $[0, T]$, let $\varphi_t(v) = \varphi(v, t)$. Then

$$\varphi_t(v) \neq p, \quad v \in \partial(Q \cap S_0) = \partial Q \cap S_0,\ 0 \le t \le T, \tag{10.29}$$

for, if $\varphi(v, t) = p$, then $\hat{\sigma}(t)v \in F^{-1}(p) = B$. This implies $G(\hat{\sigma}(t)v) \ge b_0$ by (10.1). But (10.20) and (10.1) imply that $G(\hat{\sigma}(t)v) < b_0$ for $t > 0$. Since $p \notin \partial Q$ by hypothesis, we obtain a contradiction. Thus, (10.29) holds. Consequently, the Brouwer degree $d(\varphi_t, Q \cap S_0, p)$ is defined. Since φ_t is continuous, we have

$$d(\varphi_T, Q \cap S_0, p) = d(\varphi_0, Q \cap S_0, p) = d(I, Q \cap S_0, p) = 1. \tag{10.30}$$

Hence, there is a $v \in Q$ such that $F\hat{\sigma}(T)v = p$. Consequently, $\hat{\sigma}(T)v \in F^{-1}(p) = B$. In view of (10.1), this implies

$$G(\hat{\sigma}(T)v) \ge b_0,$$

contradicting (10.26). This completes the proof. □

10.3 Some examples

The following are examples of sets that link weakly.

Example 1. Let M, N be closed subspaces such that $E = M \oplus N$ (both can be infinite-dimensional). Let

$$B_R = \{u \in E : \|u\| < R\}$$

and take $A = \partial B_R \cap N,\ B = M$. Then A links B weakly. To see this, take $\Omega = B_R \cap N,\ Q = \bar{\Omega}$. For $u \in E$, we write

$$u = v + w, \quad v \in N,\ w \in M, \tag{10.31}$$

and take F to be the projection

$$Fu = Pu = v.$$

For any finite-dimensional subspace S of E such that $FS \neq \{0\}$, take $S_0 = PS$. Since $F|_Q = I$ and $M = F^{-1}(0)$, we see by Theorem 10.2 that A links B weakly.

Example 2. We take M, N as in Example 1. Let $w_0 \neq 0$ be an element of M, and take

$$\begin{aligned} A &= \{v \in N : \|v\| \leq R\} \cup \{sw_0 + v : v \in N, s \geq 0, \|sw_0 + v\| = R\}, \\ B &= \partial B_\delta \cap M,\ 0 < \delta < R. \end{aligned}$$

Then A links B weakly. Again, we may assume that $\|w_0\| = 1$. Let

$$Q = \{sw_0 + v : v \in N, s \geq 0, \|sw_0 + v\| \leq R\}.$$

Then $A = \partial Q$ in $\tilde{N} = N \oplus \{w_0\}$. If u is given by (10.31), we define

$$Fu = v + \|w\| w_0.$$

Then F is a continuous map of E onto $\tilde{N}$, $F|_Q = I$, and $B = F^{-1}(\delta w_0)$. For any finite-dimensional subspace S of E, take $S_0 = PS \oplus \{w_0\}$. We can now apply Theorem 10.2 to conclude that A links B weakly.

Example 3. Take M, N as before and let $v_0 \neq 0$ be an element of N. We write $N = \{v_0\} \oplus N'$. We take

$$\begin{aligned} A &= \{v' \in N' : \|v'\| \leq R\} \cup \{sv_0 + v' : v' \in N', s \geq 0, \|sv_0 + v'\| = R\}, \\ B &= \{w \in M : \|w\| \geq \delta\} \cup \{sv_0 + w : w \in M, s \geq 0, \|sv_0 + w\| = \delta\}, \end{aligned}$$

where $0 < \delta < R$. Then A links B weakly. To see this, we let

$$Q = \{sv_0 + v' : v' \in N', s \geq 0, \|sv_0 + v'\| \leq R\}$$

and reason as before. For simplicity, we assume that $\|v_0\| = 1$, that E is a Hilbert space, and that the splitting $E = N' \oplus \{v_0\} \oplus M$ is orthogonal. If

$$u = v' + w + sv_0, \quad v' \in N',\ w \in M,\ s \in \mathbb{R}, \tag{10.32}$$

we define

$$\begin{aligned} F(u) &= v' + \left(s + \delta - \sqrt{\delta^2 - \|w\|^2}\right) v_0, & \|w\| \le \delta \\ &= v' + (s + \delta) v_0, & \|w\| > \delta. \end{aligned}$$

Note that $F|_Q = I$ while $F^{-1}(\delta v_0)$ is precisely the set B. For any finite-dimensional subspace S of E containing (δv_0), take $S_0 = FS \oplus \{\delta v_0\}$. Hence, we can conclude via Theorem 10.2 that A links B weakly.

Example 4. This is the same as Example 3 with A replaced by $A = \partial B_R \ \cap \ N$. The proof is the same with Q replaced by $Q = \bar{B}_R \ \cap \ N$.

Example 5. Let M, N be as in Example 1. Take $A = \partial B_\delta \cap N$, and let v_0 be any element in $\partial B_1 \cap N$. Take B to be the set of all u of the form

$$u = w + s v_0, \quad w \in M,$$

satisfying any of the following:

(a) $\|w\| \le R, \ s = 0;$

(b) $\|w\| \le R, \ s = 2R_0;$

(c) $\|w\| = R, \ 0 \le s \le 2R_0,$

where $0 < \delta < \min(R, R_0)$. Then A links B weakly. To see this, take $N = \{v_0\} \oplus N'$. Then any $u \in E$ can be written in the form (10.32). Define

$$F(u) = \left(R_0 - \max\left\{\frac{R_0}{R}\|w\|, |s - R_0|\right\}\right) v_0 + v'$$

and $Q = \bar{B}_\delta \cap N$. Then $F \in C(E, N)$ and $F|_Q = I$. Moreover, $A = F^{-1}(0)$. For any finite-dimensional subspace S of E such that $FS \neq \{0\}$, take $S_0 = PS \oplus \{v_0\}$. Hence, A links B by Theorem 10.2.

Example 6. Let M, N be as in Example 1. Let v_0 be in $\partial B_1 \cap N$ and write $N = \{v_0\} \oplus N'$. Let $A = \partial B_\delta \cap N, \ \ Q = \bar{B}_\delta \cap N$, and

$$B = \{w \in M : \|w\| \le R\} \cup \{w + s v_0 : w \in M, s \ge 0, \|w + s v_0\| = R\},$$

where $0 < \delta < R$. Then A links B weakly. To see this, write $u = w + v' + s v_0$, $w \in M, \ v' \in N', \ s \in \mathbb{R}$ and take

$$F(u) = (cR - \max\{c\|w + s v_0\|, |cR - s|\}) v_0 + v',$$

where $c = \delta/(R - \delta)$. Then F is the identity operator on Q, and $F^{-1}(0) = B$. For any finite-dimensional subspace S of E such that $FS \neq \{0\}$, take $S_0 = PS \oplus \{v_0\}$. Apply Theorem 10.2.

10.4 Some applications

In this section we apply Theorem 10.2 to semilinear boundary-value problems. Let Ω be a domain in $\mathbb{R}^n$ and let A be a self-adjoint operator in $L^2(\Omega)$ having 0 in its resolvent set.Thus, there is an interval (a, b) in its resolvent set satisfying $a < 0 < b$. Let $f(x, t)$ be a Carathéodory function on $\Omega \times \mathbb{R}$ such that

$$(10.33) \qquad |f(x,t)| \leq V(x)^2|t| + W(x)V(x), \quad x \in \Omega,\ t \in \mathbb{R},$$

and

$$(10.34) \qquad f(x,t)/t \to \alpha_\pm(x) \quad \text{as} \quad t \to \pm\infty,$$

where $V, W \in L^2(\Omega)$, and multiplication by $V(x) > 0$ is a compact operator from $D = D(|A|^{1/2})$ to $L^2(\Omega)$. Let

$$M = \int_b^\infty dE(\lambda)D, \quad N = \int_{-\infty}^a dE(\lambda)D,$$

where $\{E(\lambda)\}$ is the spectral measure of A. Then M, N are invariant subspaces for A and $D = M \oplus N$. If

$$(10.35) \qquad \alpha(u,v) = \int_\Omega (\alpha_+u^+ - \alpha_-u^-)v\,dx, \quad \alpha(u) = \alpha(u,u),$$

then we assume that

$$(10.36) \qquad \alpha(v) \geq (Av, v), \quad v \in N,$$

$$(10.37) \qquad (Aw, w) \geq \alpha(w), \quad w \in M.$$

We also assume that the only solution of

$$(10.38) \qquad Au = \alpha_+u^+ - \alpha_-u^-$$

is $u \equiv 0$, where $u^\pm = \max\{\pm u, 0\}$. We have

Theorem 10.5. *Under the above hypotheses, there is at least one solution of*

$$(10.39) \qquad Au = f(x,u), \quad u \in D(A).$$

Proof. Let

$$(10.40) \qquad a(u,v) = (Au, v), \quad a(u) = (Au, u), \quad u, v \in D,$$

and

$$(10.41) \qquad G(u) = a(u) - 2\int F(x,u)\,dx, \quad u \in D,$$

where

$$F(x,t) = \int_0^t f(x,s)ds. \tag{10.42}$$

From (10.33) it is readily verified that G is continuously differentiable on D and

$$(G'(u), v)/2 = a(u,v) - (f(u), v), \quad u, v \in D, \tag{10.43}$$

where we write $f(u)$ in place of $f(x,u)$ (cf., e.g., [112]). We note that $G'(u)$ satisfies (10.5) on $E = D$: for, if $u_k \to u$ weakly in D, then $a(u_k, v) \to a(u,v)$ for each $v \in D$, and there is a renamed subsequence such that $Vu_k \to Vu$ in $L^2(\Omega)$. Since $V(x) > 0$ in Ω, there is a renamed subsequence such that $u_k \to u$ a.e. in Ω. Thus,

$$f(x, u_k)v \to f(x,u)v \quad \text{a.e.} \tag{10.44}$$

Since

$$|f(x,u_k)v| \leq |Vu_kVv| + W|Vv|$$

and the right-hand side converges to $|VuVv| + W|Vv|$ in $L^1(\Omega)$, we see that $(f(u_k), v) \to (f(u), v)$. Thus, $G'(u)$ satisfies (10.5).

I claim that

$$G(v) \to -\infty \quad \text{as } \|v\|_D \to \infty, \quad v \in N, \tag{10.45}$$

$$G(w) \to \infty \quad \text{as } \|w\|_D \to \infty, \quad w \in M, \tag{10.46}$$

where $\|u\|_D = \||A|^{1/2}u\|$. To prove (10.45), let $\{v_k\}$ be a sequence such that $\rho_k = \|v_k\|_D \to \infty$, and let $\tilde{v}_k = v_k/\rho_k$. Then $\|\tilde{v}_k\|_D = 1$, and there is a renamed subsequence that converges weakly in D and a.e. in Ω to a function $\tilde{v} \in D$ (again using the properties of V). I claim that

$$2\int_\Omega F(x, v_k)dx/\rho_k^2 \to \alpha(\tilde{v}) \tag{10.47}$$

and

$$G(v_k)/\rho_k^2 \to -1 - \alpha(\tilde{v}) = (\|\tilde{v}\|_D^2 - 1) - (\|\tilde{v}\|_D^2 + \alpha(\tilde{v})) \leq 0. \tag{10.48}$$

To see this, note that

$$\frac{2F(x,v_k)}{\rho_k^2} = \frac{2F(x,v_k)}{v_k^2}\tilde{v}_k^2 \longrightarrow \alpha_+(\tilde{v}^+)^2 + \alpha_-(\tilde{v}^-)^2 \text{ a.e.}$$

Moreover, by (10.33),

$$|F(x,v_k)|/\rho_k^2 \leq C(|V\tilde{v}_k|^2 + W|V\tilde{v}_k|/\rho_k) \longrightarrow C|V\tilde{v}|^2$$

in $L^1(\Omega)$. This implies (10.47) and (10.48). Now, the only way the right-hand side of (10.48) can vanish is if

$$\|\tilde{v}\|_D = 1 \tag{10.49}$$

and

$$(A\tilde{v}, \tilde{v}) = \alpha(\tilde{v}). \tag{10.50}$$

Let

$$\Theta(v) = (Av, v) - \alpha(v).$$

Then

$$\Theta(v) \le 0, \quad v \in N.$$

If $\Theta(\tilde{v}) = 0$, then $\tilde{v}$ is a maximum point for Θ on N. This means that $\Theta'(\tilde{v}) = 0$, i.e., that

$$(A\tilde{v}, v) - \alpha(\tilde{v}, v) = 0, \quad v \in N.$$

Thus, $\tilde{v}$ is a solution of (10.38). However, this contradicts (10.49). Hence, the right-hand side of (10.48) must be negative. This proves (10.45). The proof of (10.46) is similar. Note that (10.46) implies

$$b_0 = \inf_M G > -\infty,$$

for if $G(w_k) \to b_0$, $\{w_k\} \subset M$, then $\|w_k\|_D \le C$ by (10.46). But then (10.33) implies that G is bounded on bounded sets in D. From (10.45) we see that there is an R such that (10.1) holds with $A = N \cap \partial B_R$, $B = M$, $Q = N \cap B_R$, and $F = P$, the orthogonal projection of D onto N, $p = 0$. If S is a finite-dimensional subspace of D such that $PS \ne \{0\}$, let $S_0 = PS$. Clearly, (10.6) is satisfied. We can now apply Theorem 10.2 to conclude that there is a sequence $\{u_k\} \subset D$ such that

$$G(u_k) \to c, \quad b_0 \le c \le a_1, \quad G'(u_k) \to 0. \tag{10.51}$$

Assume first that $\rho_k = \|u_k\|_D \to \infty$, and write $\tilde{u}_k = u_k/\rho_k$. Then $\|\tilde{u}_k\|_D = 1$. Consequently, there is a renamed subsequence such that $\tilde{u}_k \to \tilde{u}$ weakly in D and a.e. in Ω (using the properties of V). Since

$$|f(x, u_k)v|/\rho_k \le |V\tilde{u}_k Vv| + W|Vv|/\rho_k$$

and the right-hand side converges in $L^1(\Omega)$, we see by (10.34) that

$$(f(u_k), v)/\rho_k \to \alpha(\tilde{u}, v). \tag{10.52}$$

Thus,

$$(G'(u_k), v)/2\rho_k \to a(\tilde{u}, v) - \alpha(\tilde{u}, v), \quad v \in D.$$

Since this limit vanishes by (10.51), we see that $\tilde{u}$ is a solution of (10.38). Hence, $\tilde{u} \equiv 0$ by hypothesis. Let $\tilde{u}_k = \tilde{v}_k + \tilde{w}_k$, where $\tilde{v}_k \in N$, $\tilde{w}_k \in M$. Arguments similar to that given in the proof of (10.47) show that

$$(f(u_k), \tilde{v}_k)/\rho_k \to \alpha(\tilde{u}, \tilde{v})$$

and

$$(f(u_k), \tilde{w}_k)/\rho_k \to \alpha(\tilde{u}, \tilde{w}),$$

where $\tilde{u} = \tilde{v} + \tilde{w}$. Since $\|\tilde{v}_k\|_D^2 + \|\tilde{w}_k\|_D^2 = 1$, we have for a renamed subsequence

$$\begin{aligned} (G'(u_k), \tilde{v}_k)/2\rho_k &\to \gamma_1 - \alpha(\tilde{u}, \tilde{v}), \\ (G'(u_k), \tilde{w}_k)/2\rho_k &\to \gamma_2 - \alpha(\tilde{u}, \tilde{w}), \end{aligned}$$

where $\gamma_2 + \gamma_1 = 1$. By (10.51), both of these expressions must vanish. But this cannot happen if $\tilde{u} \equiv 0$. Thus, the ρ_k must be bounded, and consequently there is a renamed subsequence of $\{u_k\}$ such that $u_k \to u$ weakly in D and a.e. in Ω. As before, this implies $(f(u_k), v) \to (f(u), v)$ for $v \in D$. Hence,

$$(G'(u_k), v) \to (G'(u), v), \quad v \in D.$$

By (10.51), this limit must vanish for each v in D. We conclude that $G'(u) = 0$ and that u is a solution of (10.39). □

As another application, let $\Omega \subset \mathbb{R}^n$ and let $A \geq \lambda_0 > 0$ be a self-adjoint operator on $L^2(\Omega)$ with compact resolvent. Let $F(x, s, t)$ be a function on $\Omega \times \mathbb{R}^2$ such that

$$f(x, s, t) = \partial F/\partial t, \quad g(x, s, t) = \partial F/\partial s \tag{10.53}$$

are Carathéodory functions satisfying

$$|f(x, s, t)| + |g(x, s, t)| \leq C(|s| + |t| + 1), \quad s, t \in \mathbb{R}, \tag{10.54}$$

and

$$f(x, s, t)/\rho \to \alpha(x)\tilde{s} + \beta(x)\tilde{t}, \tag{10.55}$$

$$g(x, s, t)/\rho \to \gamma(x)\tilde{s} + \delta(x)\tilde{t} \tag{10.56}$$

as $\rho^2 = s^2 + t^2 \to \infty$, $s/\rho \to \tilde{s}$, $t/\rho \to \tilde{g}$. We wish to solve the system

$$Av = f(x, v, w), \quad Aw = g(x, v, w), \quad v, w \in D(A). \tag{10.57}$$

We assume that the only solution of

$$Av = \alpha v + \beta w, \quad Aw = \gamma v + \delta w \tag{10.58}$$

is $v \equiv w \equiv 0$ in Ω. If λ_0 is an eigenvalue of A, we assume that corresponding eigenfunctions are $\neq 0$ a.e. on Ω. Finally, we assume that

$$\beta + \gamma - \alpha - \delta \leq\not\equiv 2\lambda_0, \quad \beta + \gamma + \alpha + \delta \leq\not\equiv 2\lambda_0. \tag{10.59}$$

We have

Theorem 10.6. *Under the above assumptions, system (10.57) has a solution.*

Proof. Let $D = D(A^{1/2})$, $E = D \times D$. Then D becomes a Hilbert space with norm given by

$$\|u\|_E^2 = (Av, v) + (Aw, w), \quad u = (v, w) \in D. \tag{10.60}$$

We define

$$G(u) = (Av, w) - \int_\Omega F(x, v, w)\, dx, \quad u \in D. \tag{10.61}$$

In view of (10.54), $G \in C^1(D, \mathbb{R})$ and

$$(G'(u), h) = (Av, h_2) + (Aw, h_1) - (f(u), h_2) - (g(u), h_1) \tag{10.62}$$

for $u = (v, w) \in E$, $h = (h_1, h_2) \in E$, and we write $f(u), g(u)$ in place of $f(x, v, w)$, $g(x, v, w)$, respectively. It follows from (10.62) that $u \in E$ is a solution of (10.57) if u satisfies

$$G'(u) = 0. \tag{10.63}$$

Let N denote the subspace of E consisting of those $u = (v, w)$ for which $w = -v$ and let M consist of those for which $w = v$. Then $M = N^\perp$. We note that

$$G(v, -v) \to -\infty \text{ as } \|v\|_D \to \infty \tag{10.64}$$

and

$$G(w, w) \to +\infty \text{ as } \|w\|_D \to \infty. \tag{10.65}$$

To prove (10.64), let $\{v_k\} \subset D$ be such that $\rho_k = \|v_k\|_D \to \infty$, and take $\tilde{v}_k = v_k/\rho_k$. Since $\|\tilde{v}_k\|_D = 1$, there is a renamed subsequence such that $\tilde{v}_k \to \tilde{v}$ weakly in N, strongly in $L^2(\Omega)$, and a.e. in Ω such that

$$\begin{aligned} 2G(v_k, -v_k)/\rho_k^2 &= -2\|\tilde{v}_k\|_D^2 - 2\int_\Omega F(x, v_k, -v_k) dx/\rho_k^2 \\ &\to -2 - \int_\Omega [\gamma - \delta - \alpha + \beta]\tilde{v}^2 dx \\ &= -\int_\Omega [2\lambda_0 + \beta + \gamma - \alpha - \delta]\tilde{v}^2 dx + 2\lambda_0\|\tilde{v}\|^2 - 2. \end{aligned} \tag{10.66}$$

In view of (10.59), this is negative unless $\lambda_0\|\tilde{v}\|^2 = 1$. But this would mean that $\tilde{v} \in E(\lambda_0)$, the eigenspace of λ_0. If λ_0 is not an eigenvalue, then we conclude immediately that

$$\limsup_{\|v\|_D \to \infty} G(v, -v)/\|v\|_D^2 < 0 \tag{10.67}$$

and (10.64) holds. If λ_0 is an eigenvalue, then $\tilde{v} \neq 0$ a.e. by hypothesis. But then the last integral in (10.66) must be positive by (10.59). Again, this implies (10.67) and (10.64). A similar argument implies (10.65). Now that we have (10.64) and (10.65), we know that (10.1) holds for $A = N \cap \partial B_R$ and $B = M$ when R is sufficiently large. If we let F be the projection of E onto N, we can take $Q = N \cap B_R$ and $p = 0$. We can conclude from Theorem 10.2 that there is a sequence $\{u_k\} \subset E$ satisfying (10.2), (10.3). I claim that

$$\rho_k = \|u_k\|_E \leq C. \tag{10.68}$$

To see this, assume that $\rho_k \to \infty$, and let $\tilde{u}_k = u_k/\rho_k$. Then there is a renamed subsequence such that $\tilde{u}_k \to \tilde{u}$ weakly in E, strongly in $L^2(\Omega)$, and a.e. in Ω. If $h = (h_1, h_2) \in E$, then, by (10.62),

$$(G'(u_k), h)/\rho_k = (A\tilde{v}_k, h_2) + (A\tilde{w}_k, h_1) - (f(u_k), h_2)/\rho_k - (h(u_k), h_1)/\rho_k, \tag{10.69}$$

where $u_k = (v_k, w_k)$. Taking the limit and applying (10.54) – (10.56) we see that $\tilde{u} = (\tilde{v}, \tilde{w})$ is a solution of (10.58). By hypothesis, $\tilde{u} \equiv 0$. On the other hand, there is a renamed subsequence such that $\|\tilde{v}_k\|_D \to a$, $\|\tilde{w}_k\|_D \to b$ with $a^2 + b^2 = 1$. Moreover,

$$\begin{aligned}(G'(u_k), (\tilde{v}_k, -\tilde{v}_k))/\rho_k =& - 2\|\tilde{v}_k\|_D^2 + (f(u_k), \tilde{v}_k)/\rho_k \\ & - (g(u_k), \tilde{v}_k)/\rho_k \\ & \to - 2a^2 + \int_\Omega [\alpha + \beta - \gamma - \delta]\tilde{v}^2 dx\end{aligned} \tag{10.70}$$

and

$$\begin{aligned}(G'(u_k), (\tilde{w}_k, \tilde{w}_k))/\rho_k =& 2\|\tilde{w}_k\|_D^2 - (f(u_k), \tilde{w}_k)/\rho_k \\ & - (g(u_k), w_k)/\rho_k \\ & \to 2b^2 - \int_\Omega [\alpha + \beta + \gamma + \delta]\tilde{w}^2 dx.\end{aligned} \tag{10.71}$$

By (10.2), both these limits are 0. Since a and b cannot both vanish, the same is true of $\tilde{v}$ and $\tilde{w}$. Hence, $\tilde{u} \not\equiv 0$, and this contradiction shows that (10.68) does not hold. We can now use the usual procedures to show that $\{u_k\}$ has a renamed subsequence converging weakly in E to a function u, strongly in $L^2(\Omega)$, and a.e. in Ω. Then this limit is a solution of (10.63) and consequently of (10.57). □

For another application, let A, B be positive, self-adjoint operators on $L^2(\Omega)$ with compact resolvents, where $\Omega \subset \mathbb{R}^n$. Let $F(x, v, w)$ be a function on $\Omega \times \mathbb{R}^2$ such that

$$f(x, v, w) = \partial F/\partial v, \quad g(x, v, w) = \partial F/\partial w \tag{10.72}$$

are Carathéodory functions satisfying

$$|f(x, v, w)| + |g(x, v, w)| \leq C_0(|v| + |w| + 1), \quad v, w \in \mathbb{R}, \tag{10.73}$$

and

$$f(x, ty, tz)/t \to \alpha_+(x)v^+ - \alpha_-(x)v^- + \beta_+(x)w^+ - \beta_-(x)w^-, \tag{10.74}$$

$$g(x, ty, tz)/t \to \gamma_+(x)v^+ - \gamma_-(x)v^- + \delta_+(x)w^+ - \delta_-(x)w^- \tag{10.75}$$

as $t \to +\infty$, $y \to v$, $z \to w$, where $a^\pm = \max(\pm a, 0)$. We wish to solve the system

$$Av = -f(x, v, w), \tag{10.76}$$

$$Bw = g(x, v, w). \tag{10.77}$$

Let $\lambda_0(\mu_0)$ be the lowest eigenvalue of $A(B)$. We assume that the only solution of

$$-Av = \alpha_+v^+ - \alpha_-v^- + \beta_+w^+ - \beta_-w^-, \tag{10.78}$$

$$Bw = \gamma_+v^+ - \gamma_-v^- + \delta_+w^+ - \delta_-w^- \tag{10.79}$$

is $v = w = 0$. We have

Theorem 10.7. *Assume that eigenfunctions of λ_0 are $\neq 0$ a.e. on Ω,*

$$\alpha_\pm(x) \geq\not\equiv -\lambda_0, \quad x \in \Omega, \tag{10.80}$$

and

$$2F(x, 0, t) \leq \mu_0 t^2 + W(x), \quad x \in \Omega,\ t \in \mathbb{R}, \tag{10.81}$$

where $W(x) \in L^1(\Omega)$. Then the system (10.76), (10.77) has a solution.

Proof. Let $D = D(A^{1/2}) \times D(B^{1/2})$. Then D becomes a Hilbert space with norm given by

$$\|u\|_D^2 = (Av, v) + (Bw, w), \quad u = (v, w) \in D. \tag{10.82}$$

We define

$$G(u) = b(w) - a(v) - 2\int_\Omega F(x, v, w)\, dx, \quad u \in D, \tag{10.83}$$

where

$$a(v) = (Av, v), \quad b(w) = (Bw, w). \tag{10.84}$$

Then $G \in C^1(D, \mathbb{R})$ and

$$(G'(u), h)/2 = b(w, h_2) - a(v, h_1) - (f(u), h_1) - (g(u), h_2), \tag{10.85}$$

where we write $f(u)$, $g(u)$ in place of $f(x, v, w)$, $g(x, v, w)$, respectively. It is readily seen that the system (10.76), (10.77) is equivalent to

$$G'(u) = 0. \tag{10.86}$$

We let N be the set of those $(v, 0) \in D$ and M the set of those $(0, w) \in D$. Then M, N are orthogonal closed subspaces such that

$$D = M \oplus N. \tag{10.87}$$

If we define

$$Lu = 2(-v, w), \quad u = (v, w) \in D \tag{10.88}$$

then L is a self-adjoint, bounded operator on D. Also,

$$G'(u) = Lu + c_0(u), \tag{10.89}$$

where

$$c_0(u) = -(A^{-1} f(u), B^{-1} g(u)) \tag{10.90}$$

is compact on D. This follows from (10.73) and the fact that A and B have compact resolvents. Now, by (10.81),

$$G(0, w) \geq b(w) - \mu_0 \|w\|^2 - \int_\Omega W(x)\, dx, \quad (0, w) \in M. \tag{10.91}$$

Thus,

$$\inf_M G \geq -\int_\Omega W(x) dx. \tag{10.92}$$

On the other hand, (10.80) implies

$$\sup_{N \cap \partial B_R} G \to -\infty \text{ as } R \to \infty. \tag{10.93}$$

To see this, let $(v_k, 0)$ be any sequence in N such that $\rho_k^2 = a(v_k) \to \infty$. Then

$$G(v_k, 0)/\rho_k^2 = -a(\tilde{v}_k) - 2 \int_\Omega F(x, v_k, 0) dx / \rho_k^2, \tag{10.94}$$

where $\tilde{v}_k = v_k/\rho_k$. Note that $a(\tilde{v}_k) = 1$. Thus, there is a renamed subsequence $\tilde{v}_k \to \tilde{v}$ weakly in N, strongly in $L^2(\Omega)$, and a.e. in Ω such that

$$\begin{aligned} G(v_k, 0)/\rho_k^2 &\to -1 - \int_\Omega \{\alpha_+ (\tilde{v}^+)^2 + \alpha_- (\tilde{v}^-)^2\} dx \\ &= -\int_\Omega \{(\lambda_0 + \alpha_+)(\tilde{v}^+)^2 + (\lambda_0 + \alpha_-)(\tilde{v}^-)^2\} dx \\ &\quad + \lambda_0 \|\tilde{v}\|^2 - 1. \end{aligned}$$

This is negative unless $\lambda_0 \|\tilde{v}\|^2 = 1$. Since $a(\tilde{v}) \leq 1$, this would mean that $\tilde{v} \in E(\lambda_0)$, the eigenspace of λ_0. Thus, $\tilde{v} \neq 0$ a.e. by hypothesis. But then the integral cannot vanish by (10.80). Hence,

$$\limsup_{a(v) \to \infty} G(0, v)/a(v) < 0, \tag{10.95}$$

and (10.93) holds. Since N is an invariant subspace of L, we can apply Theorem 10.2 to conclude that there is a sequence $\{u_k\} \subset D$ such that (10.2) and (10.3) hold. Let $u_k = (v_k, w_k)$. I claim that

$$\rho_k^2 = a(v_k) + b(w_k) \leq C. \tag{10.96}$$

To see this, assume that $\rho_k \to \infty$, and let $\tilde{u}_k = u_k/\rho_k$. Then there is a renamed subsequence such that $\tilde{u}_k \to \tilde{u}$ weakly in D, strongly in $L^2(\Omega)$, and a.e. in Ω. If $h = (h_1, h_2) \in D$, then

(10.97)

$$(G'(u_k), h)/\rho_k = 2b(\tilde{w}_k, h_2) - 2a(\tilde{v}_k, h_1) - 2(f(u_k), h_1)/\rho_k - 2(g(u_k), h_2)/\rho_k.$$

Taking the limit and applying (10.73) – (10.75), we see that $\tilde{u} = (\tilde{v}, \tilde{w})$ is a solution of (10.78) (10.79). Hence, $\tilde{u} = 0$ by hypothesis. On the other hand, since $a(\tilde{v}_k) + b(\tilde{w}_k) = 1$, there is a renamed subsequence such that $a(\tilde{v}_k) \to \tilde{a}$, $b(\tilde{w}_k) \to \tilde{b}$ with $\tilde{a} + \tilde{b} = 1$. Thus, by (10.74), (10.75), and (10.85),

$$\begin{aligned}(G'(u_k), (\tilde{v}_k, 0))/2\rho_k &= -a(\tilde{v}_k) - (f(u_k), \tilde{v}_k)/\rho_k \\ &\to -\tilde{a} - \int_\Omega (\alpha_+\tilde{v}^+ - \alpha_-\tilde{v}^- + \beta_+\tilde{w}^+ - \beta_-\tilde{w}^-)\tilde{v}\,dx\end{aligned} \tag{10.98}$$

and

$$\begin{aligned}(G'(u_k), (0, \tilde{w}_k))/2\rho_k &= b(\tilde{w}_k) - (g(u_k), \tilde{w}_k)/\rho_k \\ &\to \tilde{b} - \int_\Omega (\gamma_+\tilde{v}^+ - \gamma_-\tilde{v}^- + \delta_+\tilde{w}^+ - \delta_-\tilde{w}^-)\tilde{w}\,dx.\end{aligned} \tag{10.99}$$

Thus, by (10.42),

$$\tilde{a} = -\int_\Omega (\alpha_+\tilde{v}^+ - \alpha_-\tilde{v}^- + \beta_+\tilde{w}^+ - \beta_-\tilde{w}^-)\tilde{v}\,dx \tag{10.100}$$

and

$$\tilde{b} = \int_\Omega (\gamma_+\tilde{v}^+ - \gamma_-\tilde{v}^- + \delta_+\tilde{w}^+ - \delta_-\tilde{w}^-)\tilde{w}\,dx. \tag{10.101}$$

Since one of the two numbers $\tilde{a}, \tilde{b}$ is not zero, we see that we cannot have $\tilde{u} \equiv 0$. This contradiction proves (10.96). Once this is known, we can use the usual procedures to show that there is a renamed subsequence such that $u_k \to u$ in D, and u satisfies (10.86). $\square$

Corollary 10.8. *In Theorem 10.7 we can replace (10.80), (10.81) with*

$$2F(x, v, 0) \geq -\lambda_0 v^2 - W(x), \quad x \in \Omega,\ v \in \mathbb{R}, \tag{10.102}$$

provided eigenfunctions of μ_0 *are* $\neq 0$ *a.e. in* Ω *and*

$$\delta_\pm(x) \leq\not\equiv \mu_0. \tag{10.103}$$

Proof. We interchange the roles of M and N in the proof of Theorem 10.7. $\square$

Theorem 10.9. *If λ_0 is simple and the eigenfunctions of λ_0 and μ_0 are bounded and $\neq 0$ a.e. in Ω, we can replace (10.81) in Theorem 10.7 with*

$$2F(x,t,0) \geq -\lambda_1 t^2, \quad t \in \mathbb{R}, \tag{10.104}$$

$$2F(x,s,t) \leq \mu_0 t^2 - \lambda_0 s^2, \quad |t|+|s| \leq \delta, \tag{10.105}$$

where λ_1 is the next eigenvalue of A and $\delta > 0$. Moreover, system (10.76), (10.77) has a nontrivial solution.

Proof. Let N' be the orthogonal complement of $N_0 = \{\varphi_0\}$ in N. Then $N = N' \oplus N_0$ and

$$G(v',0) = -a(v') - 2\int_\Omega F(x,v',0)\,dx \leq -a(v') + \lambda_1 \|v'\|^2 \leq 0, \quad v' \in N', \tag{10.106}$$

and

$$G(v,0) \to -\infty \text{ as } a(v) \to \infty \tag{10.107}$$

by (10.80) [cf. (10.93)]. Let M_0 be the subspace of M spanned by the eigenfunctions of B corresponding to μ_0, and let M' be its orthogonal complement in M. Since N_0 and M_0 are contained in $L^\infty(\Omega)$, there is a positive constant ρ such that

$$a(y) \leq \rho^2 \Rightarrow \|y\|_\infty \leq \delta/4, \quad y \in N_0, \tag{10.108}$$

$$b(h) \leq \rho^2 \Rightarrow \|h\|_\infty \leq \delta/4, \quad h \in M_0, \tag{10.109}$$

where δ is the number given in (10.105). If

$$a(y) \leq \rho^2, \quad b(w) \leq \rho^2, \quad |y(x)| + |w(x)| \geq \delta, \tag{10.110}$$

we write $w = h + w'$, $h \in M_0$, $w' \in M'$, and

$$\delta \leq |y(x)| + |w(x)| \leq |y(x)| + |h(x)| + |w'(x)| \leq (\delta/2) + |w'(x)|. \tag{10.111}$$

Thus,

$$|y(x)| + |h(x)| \leq \delta/2 \leq |w'(x)| \tag{10.112}$$

and

$$|y(x)| + |w(x)| \leq 2|w'(x)|. \tag{10.113}$$

Now, by (10.105) and (10.113),

$$
\begin{aligned}
G(y,w) =& b(w) - a(y) - 2\int_\Omega F(x,y,w)dx \\
\geq& b(w) - a(y) - \int_{|y|+|w|<\delta} \{\mu_0 w^2 - \lambda_0 y^2\}dx \\
& - c_0 \int_{|y|+|w|>\delta} (|y|+|w|+1)dx \\
\geq& b(w) - a(y) - \mu_0\|w\|^2 + \lambda_0\|y\|^2 - c_1 \int_{2|w'|>\delta} |w'|^4 dx \\
\geq& b(w') - \mu_0\|w'\|^2 - c_2 b(w')^2 \\
\geq& \left(1 - \frac{\mu_0}{\mu_1} - c_2 b(w')\right) b(w')
\end{aligned}
$$

when

$$a(y) \leq \rho^2, \quad b(w) \leq \rho^2,$$

where μ_1 is the next eigenvalue of B after μ_0. If we reduce ρ accordingly, we can find a positive constant ν such that

$$G(y,w) \geq \nu b(w'), \quad a(y) \leq \rho^2, b(w) \leq \rho^2. \tag{10.114}$$

I claim that either (a) (15.48), (15.49) has a nontrivial solution or (b) there is an $\epsilon > 0$ such that

$$G(y,w) \geq \epsilon, \quad a(y) + b(w) = \rho^2. \tag{10.115}$$

To see this, suppose (10.115) did not hold. Then there would be a sequence $\{y_k, w_k\}$ such that $a(y_k) + b(w_k) = \rho^2$ and $G(y_k, w_k) \to 0$. If we write $w_k = w_k' + h_k$, $w_k' \in M'$, $h_k \in M_0$, then (10.114) tells us that $b(w_k') \to 0$. Thus, $a(y_k) + b(h_k) \to \rho^2$. Since N_0, M_0 are finite-dimensional, there is a renamed subsequence such that $y_k \to y$ in N_0 and $h_k \to h$ in M_0. By (10.108) and (10.109), $\|y\|_\infty \leq \delta/4$ and $\|h\|_\infty \leq \delta/4$. Consequently, (10.105) implies

$$2F(x,y,h) \leq \mu_0 h^2 - \lambda_0 y^2. \tag{10.116}$$

Since

$$G(y,h) = b(h) - a(y) - 2\int_\Omega F(x,y,h)\,dx = 0, \tag{10.117}$$

we have

$$\int_\Omega \{2F(x,y,h) + \lambda_0 y^2 - \mu_0 h^2\}dx = 0. \tag{10.118}$$

In view of (10.116), this implies

$$2F(x, y, h) \equiv \mu_0 h^2 - \lambda_0 y^2. \tag{10.119}$$

For $\zeta \in C_0^\infty(\Omega)$ and $t > 0$ small, we have

$$2[F(x, y + t\zeta, h) - F(x, y, h)]/t \le -\lambda_0[(y + t\zeta)^2 - y^2]/t. \tag{10.120}$$

Taking $t \to 0$, we have

$$f(x, y, h)\zeta \le -\lambda_0 y\zeta. \tag{10.121}$$

Since this is true for all $\zeta \in C_0^\infty(\Omega)$, we have

$$f(x, y, h) = -\lambda_0 = -Ay. \tag{10.122}$$

Similarly,

$$2[F(x, y, h + t\zeta) - F(x, y, h)]/t \le \mu_0[(h + t\zeta)^2 - h^2]/t, \tag{10.123}$$

and, consequently,

$$g(x, y, h)\zeta \le \mu_0 h\zeta \tag{10.124}$$

and

$$g(x, y, h) = \mu_0 h = Bh. \tag{10.125}$$

We see from (15.89) and (10.125) that (10.76), (10.77) has a nontrivial solution. Thus, we may assume that (10.115) holds. But then we can combine it with (10.106) and (10.107) to conclude from Theorem 10.2 that there is a sequence $\{u_k\} \subset D$ such that (10.3) holds with $c \ge \epsilon$. Arguing as in the proof of Theorem 10.7, we see that there is a $u \in D$ such that $G(u) = c$, $G'(u) = 0$. Since $c \ne 0$ and $G(0) = 0$, we see that $u \ne 0$, and the proof is complete. □

Theorem 10.10. *Assume that λ_0 is simple and the eigenfunctions corresponding to λ_0 and μ_0 are bounded and $\ne 0$ a.e. in Ω. Assume (10.80), (10.105) and*

$$2F(x, 0, t) \le \mu(x)t^2, \quad x \in \Omega, \ t \in \mathbb{R}, \tag{10.126}$$

where

$$\mu(x) \le\not\equiv \mu_0, \quad x \in \Omega. \tag{10.127}$$

Then (10.76), (10.77) has a nontrivial solution.

Proof. As in the proof of Theorem 10.7, (10.80) implies (10.93). As in the proof of Theorem 10.9, (10.105) implies (10.115) for $y \in N_0$, $w \in M$ unless (10.76), (10.77) has a nontrivial solution. Thus, we may assume that (10.115) holds. Next we note that there is a $\nu > 0$ such that

$$G(0, w) \ge \nu b(w), \quad w \in M. \tag{10.128}$$

Assuming this for the moment, we see that

$$\inf_B G \geq \varepsilon_1 > 0, \tag{10.129}$$

where

$$B = \{w \in M : b(w) \geq \rho^2\} \cup \{u = (s\varphi_0, w) : s \geq 0, w \in M\}, \tag{10.130}$$

and $\epsilon_1 = \min\{\epsilon, \nu\rho^2\}$. By (10.93), there is an $R > \rho$ such that

$$\sup_A G \leq 0, \tag{10.131}$$

where $A = N \cap \partial B_R$. We can now apply Theorem 10.2 to conclude that there is a sequence in D satisfying (10.3). As in the proof of Theorem 10.7, this leads to a solution of (10.86) satisfying $G(u) = c \geq \epsilon_1$. Hence, $u \neq 0$, and we have a nontrivial solution of (10.76), (10.77). It therefore remains only to prove (10.128). Clearly, $\nu \geq 0$. If $\nu = 0$, then there is a sequence $\{w_k\} \subset M$ such that

$$G(0, w_k) \to 0, \quad b(w_k) = 1. \tag{10.132}$$

Thus, there is a renamed subsequence such that $w_k \to w$ weakly in M, strongly in N, and a.e. in Ω. Consequently,

$$G(0, w_k) \geq 1 - \int_\Omega \mu(x) w_k^2 dx \geq \int_\Omega [\mu_0 - \mu(x)] w_k^2 dx \to 0 \tag{10.133}$$

and

$$1 = \int_\Omega \mu(x) w^2 dx \leq \mu_0 \|w\|^2 \leq b(x) \leq 1, \tag{10.134}$$

which means that we have equality throughout. It follows that we must have $w \in E(\mu_0)$, the eigenspace of μ_0. Since $w \not\equiv 0$, we have $w \neq 0$ a.e. But

$$\int_\Omega [\mu_0 - \mu(x)] w^2 dx = 0 \tag{10.135}$$

implies that the integrand vanishes identically on Ω, and consequently $\mu(x) \equiv \mu_0$, violating (10.127). This establishes (10.128) and completes the proof of the theorem. □

10.5 Notes and remarks

The concept of weak linking was begun in [19] and [78]. Further work in this direction was carried out in [106]. The main thrust was to consider the case when G can be written in the form

$$G(u) = \frac{1}{2}(Lu, u) + b(u) \tag{10.136}$$

where L is a self-adjoint operator and $b'(u)$ is weakly continuous and E is a Hilbert space. With this method, very few sets have been found to link. The results of [19], [78], and [122] do not require E to be separable. However, they require G to be of the form (10.136) with $b'(u)$ compact. In [78] the compactness of $b'(u)$ is replaced by the assumption that it be weak-to-weak continuous, and $b(u)$ itself is required to be bounded from below and be weakly lower semicontinuous. A theorem is given in [78] that requires only (10.5) but also requires G to be τ-upper semi-continuous (the τ-topology is specially constructed to accommodate the splitting of E into subspaces). In all of the results mentioned only two examples of linking sets A, B are given. The presentation given here is from [117], [118], [140]. It has several advantages. It does not require G to be of the form (10.136) [which indeed satisfies (10.5)]. It does not require hypotheses on each of an exhausting sequence of finite-dimensional subspaces. Moreover, all sets A, B known to link when one of the subspaces M, N is finite-dimensional will link now as well.

Chapter 11

Fučík Spectrum: Resonance

11.1 Introduction

Let Ω be a bounded domain in $\mathbb{R}^n$ and let $A \geq \lambda_0 > 0$ be a self-adjoint operator on $L^2(\Omega)$ with compact resolvent and eigenvalues

$$0 < \lambda_0 < \lambda_1 < \cdots < \lambda_k < \cdots . \tag{11.1}$$

If $f(x, t)$ is a Carathéodory function on $\Omega \times \mathbb{R}$, then the semilinear problem

$$Au = f(x, u), \quad u \in D(A) \tag{11.2}$$

is said to have asymptotic resonance at infinity if

$$f(x, t)/t \to \lambda_k \text{ as } |t| \to \infty, \tag{11.3}$$

where λ_k is one of the eigenvalues of A. Interest in resonant problems began with the pioneering work of Landesman and Lazer [79] and has continued until the present because such problems are more difficult to solve than nonresonant problems. The reason is that for $|u(x)|$ large, (11.2) approximates the eigenvalue problem

$$Au = \lambda_k u \tag{11.4}$$

with its inherent instabilities. Even if (11.3) does not occur, but

$$\begin{aligned} f(x, t)/t \quad &\to \quad a \text{ a.e. as } t \to -\infty \\ &\to \quad b \text{ a.e. as } t \to +\infty, \end{aligned} \tag{11.5}$$

one encounters the same difficulties if

$$Au = bu^+ - au^-, \quad u^\pm = \max\{\pm u, 0\} \tag{11.6}$$

has a nontrivial solution. In fact, the difficulties are compounded because there is no eigenspace with which to work in this case. We call the set Σ of those $(a, b) \in \mathbb{R}^2$ for

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which (11.6) has nontrivial solutions the Fučík spectrum of A. For each ℓ, it was shown in [111, 122] that in the square $[\lambda_{\ell-1}, \lambda_{\ell+1}]^2$ there are decreasing curves $C_{\ell,1}, C_{\ell,2}$ (which may coincide) passing through the point $(\lambda_\ell, \lambda_\ell)$ such that all points above or below both curves in the square are not in Σ. We denote the lower curve by $C_{\ell,1}$ and the upper curve by $C_{\ell,2}$. Further discussions concerning the Fučík spectrum can be found in Chapter 14.

In the present chapter we shall study resonance problems with respect to the Fučík spectrum. We shall allow (a, b) to be on either of the curves $C_{\ell,j}$ in $Q_\ell = (\lambda_{\ell-1}, \lambda_{\ell+1})^2$. We do not presently consider the case when (a, b) is a point between the curves (when there are points in Σ in that region). Such points will be discussed in Chapter 14. We have

Theorem 11.1. *Let $f(x, t)$ be a Carathéodory function such that*

$$|f(x, t)| \leq C(|t| + 1), \quad x \in \Omega,\ t \in \mathbb{R}, \tag{11.7}$$

and (11.6) has a nontrivial solution with $(a, b) \in C_{\ell,1}$. Assume that

$$\lambda_{\ell-1} t^2 \leq 2F(x, t) + W_1(x), \quad x \in \Omega,\ t \in \mathbb{R}, \tag{11.8}$$

and

$$H(x, t) \leq W_0(x), \quad x \in \Omega,\ t \in \mathbb{R}, \tag{11.9}$$

where $W_i(x) \in L^1(\Omega)$,

$$F(x, t) := \int_0^t f(x, s)ds, \tag{11.10}$$

and

$$H(x, t) := 2F(x, t) - tf(x, t). \tag{11.11}$$

Assume further that

$$H(x, t) \to H_\pm(x) \text{ a.e. as } t \to \pm\infty \tag{11.12}$$

and

$$\int_{u>0} H_+(x)dx + \int_{u<0} H_-(x)dx + \int_{u=0} W_0(x)dx < -B_1 = -\int_\Omega W_1(x)dx \tag{11.13}$$

whenever u is a nontrivial solution of (11.6). Then (11.2) has a solution.

Theorem 11.2. *If $f(x, t)$ satisfies (11.7) and (11.6) has a nontrivial solution with $(a, b) \in C_{\ell,2}$, assume that*

$$2F(x, t) \leq \lambda_{\ell+1} t^2 + W_1(x), \quad x \in \Omega,\ t \in \mathbb{R}, \tag{11.14}$$

$$H(x, t) \geq -W_0(x), \quad x \in \Omega,\ t \in \mathbb{R}, \tag{11.15}$$

and

$$\int_{u>0} H_+(x)dx + \int_{u<0} H_-(x)dx - \int_{u=0} W_0(x)dx > B_1 \tag{11.16}$$

for all nontrivial solutions u of (11.6). Then (11.2) has a solution.

Note that (11.13) is automatically satisfied if

$$H(x,t) \to -\infty \text{ a.e. as } |t| \to \infty \tag{11.17}$$

and (11.16) is automatically satisfied if

$$H(x,t) \to +\infty \text{ a.e. as } |t| \to \infty. \tag{11.18}$$

11.2 The curves

In this section we shall describe the curves $C_{\ell,1}$ and $C_{\ell,2}$ and discuss some of their properties. For each fixed positive integer ℓ, we let N_ℓ denote the subspace of E spanned by the eigenfunctions corresponding to $\lambda_0, \dots, \lambda_\ell$, and we let $M_\ell = N_\ell^\perp \cap D$, where $D = D(A^1/2)$. Then $D = M_\ell \oplus N_\ell$. For $a, b \in \mathbb{R}$, we define

$$I(u,a,b) = (Au,u) - a\|u^-\|^2 - b\|u^+\|^2, \quad u \in D, \tag{11.19}$$

$$M_\ell(a,b) = \inf_{\substack{w \in M_\ell \\ \|w\|=1}} \sup_{v \in N_\ell} I(v+w,a,b), \tag{11.20}$$

$$m_\ell(a,b) = \sup_{\substack{v \in N_\ell \\ \|v\|=1}} \inf_{w \in M_\ell} I(v+w,a,b), \tag{11.21}$$

$$\nu_\ell(a) = \sup\{b : M_\ell(a,b) \geq 0\}, \tag{11.22}$$

$$\mu_\ell(a) = \inf\{b : m_\ell(a,b) \leq 0\}. \tag{11.23}$$

We have

Lemma 11.3. *[111, 122] In the square Q_ℓ the functions $\mu_\ell(a), \nu_{\ell-1}(a)$ have the following properties:*

(a) $\mu_\ell(\lambda_\ell) = \nu_{\ell-1}(\lambda_\ell) = \lambda_\ell$;

(b) $\mu_\ell(a), \nu_{\ell-1}(a)$ *are continuous and strictly decreasing;*

(c) $(a, \mu_\ell(a))$ *and* $(a, \nu_{\ell-1}(a))$ *are in* Σ;

(d) *if* $(a,b) \in Q_\ell$ *and* $b > \mu_\ell(a)$, *then* $(a,b) \notin \Sigma$;

(e) *if* $(a,b) \in Q_\ell$ *and* $b < \nu_{\ell-1}(a)$, *then* $(a,b) \notin \Sigma$.

It follows from Lemma 11.3 that $\nu_{\ell-1}(a) \leq \mu_\ell(a)$ in Q_ℓ. If $\nu_{\ell-1}(a) < \mu_\ell(a)$, this lemma does not cover those points in Q_ℓ for which

$$\nu_{\ell-1}(a) < b < \mu_\ell(a). \tag{11.24}$$

We let $C_{\ell,1}$ be the lower curve $b = \nu_{\ell-1}(a)$ and we take $C_{\ell,2}$ to be the upper curve $b = \mu_\ell(a)$. We shall need

Lemma 11.4. *[111, 122] If $(a, b) \in Q_\ell$, $w \in M_{\ell-1}$, $v_1, v_2 \in N_{\ell-1}$, and*

$$I'(w + v_j, a, b) \perp N_{\ell-1}, \quad j = 1, 2, \tag{11.25}$$

then $v_1 = v_2$.

Lemma 11.5. *[111, 122] If $(a, b) \in Q_\ell$, $v \in N_\ell$, $w_1, w_2 \in M_\ell$, and*

$$I'(v + w_j, a, b) \perp M_\ell, \quad j = 1, 2, \tag{11.26}$$

then $w_1 = w_2$.

Lemma 11.6. *[111, 122] If $(a, b) \in Q_\ell$, then, for each $w \in M_{\ell-1}$, there is a $v_0 \in N_{\ell-1}$ such that*

$$I(w + v_0, a, b) = \sup_{v \in N_{\ell-1}} I(w + v, a, b). \tag{11.27}$$

Lemma 11.7. *[111, 122] If $(a, b) \in Q_\ell$, then, for each $v \in N_\ell$, there is a $w_0 \in M_\ell$ such that*

$$I(v + w_0, a, b) = \inf_{w \in M_\ell} I(v + w, a, b). \tag{11.28}$$

Lemma 11.8. *If $(a, b) \in C_{\ell,1}$, then*

$$I(u, a, b) \geq 0, \quad u \in S_{\ell,1}, \tag{11.29}$$

where

$$S_{\ell,1} = \{u \in E : I'(u, a, b) \perp N_{\ell-1}\}. \tag{11.30}$$

Proof. If $(a, b) \in C_{\ell,1}$, then $b = \nu_{\ell-1}(a)$. Thus, by (11.22),

$$M_{\ell-1}(a, b) = 0. \tag{11.31}$$

This implies by (11.20) that

$$\sup_{v \in N_{\ell-1}} I(v + w, a, b) \geq 0, \quad w \in M_{\ell-1}. \tag{11.32}$$

Let u_0 be any element in $S_{\ell,1}$. Then $u_0 = w_0 + v_0$, $w_0 \in M_{\ell-1}$, $v_0 \in N_{\ell-1}$, and

$$I'(w_0 + v_0, a, b) \perp N_{\ell-1} \tag{11.33}$$

by (11.30). In view of Lemma 11.6, there is a $v_1 \in N_{\ell-1}$ such that

$$I(w_0 + v_1, a, b) = \sup_{v \in N_{\ell-1}} I(w_0 + v, a, b). \tag{11.34}$$

From this and (11.32), it follows that

$$I'(w_0 + v_1, a, b) \perp N_{\ell-1} \tag{11.35}$$

and

$$I(w_0 + v_1, a, b) \geq 0. \tag{11.36}$$

But (11.33) and (11.35) imply via Lemma 11.4 that $v_1 = v_0$. Thus,

$$I(u_0, a, b) \geq 0,$$

and the lemma is proved. $\square$

Lemma 11.9. *If $(a, b) \in C_{\ell,2}$, then*

$$I(u, a, b) \leq 0, \quad u \in S_{\ell,2}, \tag{11.37}$$

where

$$S_{\ell,2} = \{u \in E : I'(u, a, b) \perp M_\ell\}. \tag{11.38}$$

Proof. If $(a, b) \in C_{\ell,2}$, then $b = \mu_\ell(a)$. Consequently, by (11.23),

$$m_\ell(a, b) = 0. \tag{11.39}$$

By (11.21), this implies

$$\inf_{w \in M_\ell} I(v + w, a, b) \leq 0, \quad v \in N_\ell. \tag{11.40}$$

Let u_0 be any element in $S_{\ell,2}$ and write $u_0 = v_0 + w_0$ with $v_0 \in N_\ell$, $w \in M_\ell$. Then

$$I'(v_0 + w_0, a, b) \perp M_\ell. \tag{11.41}$$

By Lemma 11.7, there is a $w_1 \in M_\ell$ such that

$$I(v_0 + w_1, a, b) = \inf_{w \in M_\ell} I(v_0 + w, a, b). \tag{11.42}$$

Consequently,

$$I'(v_0 + w_1, a, b) \perp M_\ell \tag{11.43}$$

and

$$I(v_0 + w_1, a, b) \leq 0. \tag{11.44}$$

If we now apply Lemma 11.5, we see that $w_1 = w_0$. Thus, $I(u_0, a, b) \leq 0$, and the proof is complete. $\square$

Lemma 11.10. *$N_{\ell-1} \cap \partial B_R$ links $S_{\ell,1}$ [hm] for each $R > 0$.*

Proof. We suppress the subscript $\ell - 1$. Let P be the orthogonal projection of D onto N, and define

$$F(u) = PI'(u), \quad u \in D. \tag{11.45}$$

I claim that the restriction F_0 of F to N is a homeomorphism of N onto N. If this is so, then the image of $N \cap \bar{B}_R$ under F_0 is the closure of a bounded, open set. Moreover, this open set contains the point 0 since $F_0(0) = 0$ and 0 is an interior point of $N \cap B_R$. Proposition 3.10 can then be used to show that $N \cap \partial B_R$ links $F^{-1}(0) = S_{\ell,1}$. Clearly, F_0 is a continuous map of N into itself. It is surjective. To see this, let h be any element of N and take

$$G_0(v) = I(v, a, b) - (h, v). \tag{11.46}$$

I claim that

$$G_0(v) \to -\infty \text{ as } \|v\| \to \infty, \quad v \in N. \tag{11.47}$$

Assuming this for the moment, we see that $G_0(v)$ has a maximum on N. At a point of maximum we have $G_0'(v) \perp N$, producing a solution of

$$PI'(v, a, b) = h. \tag{11.48}$$

Moreover, if

$$PI'(v_0, a, b) = h_0, \quad PI'(v_1, a, b) = h_1, \tag{11.49}$$

then

$$(h, v) = (Av, v) + a(v_1^- - v_0^-, v) - b(v_1^+ - v_0^+, v) \tag{11.50}$$

where $h = h_1 - h_0$, $v = v_1 - v_0$. Thus,

$$\begin{aligned}(b - \lambda_{\ell-1})(v_1^+ - v_0^+, v) - (a - \lambda_{\ell-1})(v_1^- - v_0^-, v),& \\ + [\lambda_{\ell-1}\|v\|^2 - (Av, v)] = -(h, v).&\end{aligned} \tag{11.51}$$

Since $a, b > \lambda_{\ell-1}$, the left-hand side of (11.51) is positive for all $v \not\equiv 0$. Consequently, there is a $\delta > 0$ such that

$$\delta\|v\|^2 \le \|h\|\|v\| \tag{11.52}$$

showing not only that F_0 is injective but that F_0^{-1} is continuous. To prove (11.47), let $\{v_k\} \subset N$ be a sequence such that $\rho_k^2 = (Av_k, v_k) \to \infty$, and let $\tilde{v}_k = v_k/\rho_k$. Then $\|\tilde{v}_k\|_D^2 = (A\tilde{v}_k, \tilde{v}_k) = 1$ and there is a subsequence such that $\tilde{v}_k \to \tilde{v}$ in D. We have

$$\begin{aligned} G_0(v_k)/\rho_k^2 &= I(\tilde{v}_k, a, b) - (h, \tilde{v}_k)/\rho_k \\ &\to I(\tilde{v}, a, b) \\ &= [\|\tilde{v}\|_D^2 - \lambda_{\ell-1}\|\tilde{v}\|^2] \\ &\quad +(\lambda_{\ell-1} - a)\|\tilde{v}^-\|^2 \\ &\quad +(\lambda_{\ell-1} - b)\|\tilde{v}^+\|^2. \end{aligned} \tag{11.53}$$

Since $\|\tilde{v}\|_D = 1$, the right-hand side of (11.53) is negative. This gives (11.47) and completes the proof of the lemma. □

The following can be proved in the same way.

Lemma 11.11. *$M_\ell \cap \partial B_R$ links $S_{\ell,2}$ [hm] for each $R > 0$.*

11.3 Existence

In this section we give the proofs of Theorems 11.1 and 11.2. Let

$$p(x,t) = f(x,t) + at^- - bt^+, \quad t^\pm = \max\{\pm t, 0\}, \tag{11.54}$$

and

$$P(x,t) = \int_0^t p(x,s)\,ds. \tag{11.55}$$

Under hypothesis (11.7), it is readily checked [122] that the functional

$$G(u) = \|u\|_D^2 - 2\int_\Omega F(x,u)\,dx = I(u,a,b) - 2\int_\Omega P(x,u)\,dx \tag{11.56}$$

is in C^1 on D with

$$(G'(u), v) = 2(u,v)_D - 2(f(\cdot,u), v) = (I'(u,a,b), v) - 2(p(\cdot,u), v) \tag{11.57}$$

and

$$(I'(u,a,b), v)/2 = (u,v)_D + a(u^-, v) - b(u^+, v). \tag{11.58}$$

From (11.5) and (11.54) we see that

$$p(x,t)/t \to 0, \quad 2P(x,t)/t^2 \to 0 \text{ as } |t| \to \infty. \tag{11.59}$$

This and the fact that

$$\partial(F/t^2)/\partial t = \partial(P/t^2)/\partial t = -t^{-3}H \tag{11.60}$$

imply that

$$\begin{aligned} P(x,t) &= t^2 \int_t^\infty s^{-3} H(x,s)ds, \quad t > 0 \\ &= -t^2 \int_{-\infty}^t s^{-3} H(x,s)ds, \quad t < 0. \end{aligned} \tag{11.61}$$

Proof of Theorem 11.1. By (11.9) and (11.61), we have

$$2P(x,t) \le W_0(x), \quad x \in \Omega, \ t \in \mathbb{R}. \tag{11.62}$$

In view of Lemma 11.8, this implies

$$G(u) \ge -2\int_\Omega P(x,u)dx \ge -B_0 = -\int_\Omega W_0(x)dx, \quad u \in S_{\ell,1}. \tag{11.63}$$

I claim that

$$G(v) \to -\infty \text{ as } \|v\| \to \infty, \quad v \in N_{\ell-1}. \tag{11.64}$$

Assume this for the moment. Then there is an $R > 0$ sufficiently large that (3.4) holds with $A = N_{\ell-1} \cap \partial B_R$ and $B = S_{\ell,1}$. Moreover, for this choice of A and B, A links B [hm] by Lemma 11.10. We can now apply Theorem 3.4 to conclude that there is a sequence $\{u_k\} \subset D$ such that

$$G(u_k) \to c, \quad (1 + \|u_k\|_D)G'(u_k) \to 0, \tag{11.65}$$

and from (11.56) we readily estimate c by

$$-B_0 \le c \le B_1. \tag{11.66}$$

From (11.65) we find

$$I(u_k, a, b) - 2\int_\Omega P(x, u_k)dx \to c, \tag{11.67}$$

$$I(u_k, a, b) - (p(\cdot, u_k), u_k) \to 0, \tag{11.68}$$

$$(I'(u_k, a, b), v) - 2(p(\cdot, u_k), v) \to 0, \quad v \in D. \tag{11.69}$$

Assume that

$$\rho_k = \|u_k\|_D \to \infty, \tag{11.70}$$

and let $\tilde{u}_k = u_k/\rho_k$. Then $\|\tilde{u}_k\|_D = 1$. Thus, there is a renamed subsequence such that $\tilde{u}_k \to \tilde{u}$ weakly in D, strongly in $L^2(\Omega)$, and a.e. in Ω. As a consequence, (11.59) and (11.68) imply

$$1 = a\|\tilde{u}^-\|^2 + b\|\tilde{u}^+\|^2. \tag{11.71}$$

This shows that $\tilde{u} \not\equiv 0$. Moreover, (11.69) implies

$$I'(\tilde{u}, a, b) = 0. \tag{11.72}$$

Thus, $\tilde{u}$ is a solution of (11.5) and satisfies

$$I(\tilde{u}, a, b) = 0. \tag{11.73}$$

If we combine this with (11.71), we see that $\|\tilde{u}\|_D = 1$. Consequently, $\tilde{u}_k \to \tilde{u}$ strongly in D. Combining (11.67) and (11.68), we obtain

$$\int_\Omega H(x, u_k)\, dx \to -c. \tag{11.74}$$

Let $\Omega_\pm = \{x \in \Omega : \pm\tilde{u}(x) > 0\}, \Omega_0 = \{x \in \Omega : \tilde{u}(x) = 0\}$. Then $u_k(x) \to \pm\infty$ for $x \in \Omega_\pm$. By (11.9) and (11.13),

$$\begin{aligned} -c &= \lim \int_\Omega H(x, u_k)\, dx \\ &\leq \int_{\Omega_+} H_+(x)\, dx \\ &\quad + \int_{\Omega_-} H_-(x)\, dx \\ &\quad + \int_{\Omega_0} W_0(x)\, dx \\ &< -B_1, \end{aligned}$$

contradicting (11.66). Thus, (11.70) does not hold. Once we know that the sequence $\{u_k\}$ is bounded in D, we can use standard techniques [122] to obtain a solution of

$$G(u) = c, \quad -B_0 \leq c \leq B_1, \quad G'(u) = 0, \tag{11.75}$$

which is a solution of (11.2).

It thus remains to prove (11.64). Let $\{v_k\} \subset N_{\ell-1}$ be a sequence such that $\rho_k = \|v_k\|_D \to \infty$. Let $\tilde{v}_k = v_k/\rho_k$. Then $\|\tilde{v}_k\|_D = 1$ and there is a renamed subsequence such that $\tilde{v}_k \to \tilde{v}$ in D and a.e. in Ω. Since

$$|F(x,t)| \leq C(t^2 + |t|) \tag{11.76}$$

by (11.7), we have

$$|F(x, v_k)/\rho_k^2| \leq C(\tilde{v}_k^2 + |\tilde{v}_k|/\rho_k) \tag{11.77}$$

and, consequently,

$$2\int_\Omega F(x, v_k) dx/\rho_k^2 \to a\|\tilde{v}^-\|^2 + b\|\tilde{v}^+\|^2 \tag{11.78}$$

by (11.6). This means that

$$\begin{aligned} G(v_k)/\rho_k^2 &= \|\tilde{v}_k\|_D^2 - 2\int_\Omega F(x, v_k) dx/\rho_k^2 \\ &\to \|\tilde{v}\|_D^2 - a\|\tilde{v}^-\|^2 - b\|\tilde{v}^+\|^2 \\ &\leq (\lambda_{\ell-1} - a)\|\tilde{v}^-\|^2 + (\lambda_{\ell-1} - b)\|\tilde{v}^+\|^2. \end{aligned}$$

Since both a and b are greater than $\lambda_{\ell-1}$, this will always be positive since $\tilde{v} \not\equiv 0$. Thus, (11.64) holds, and the proof of Theorem 11.1 is complete. □

Proof of Theorem 11.2. By (11.15) and (11.61) we have

$$-2P(x,t) \le W_0(x), \quad x \in \Omega,\ t \in \mathbb{R}. \tag{11.79}$$

Consequently,

$$G(u) \le -2\int_\Omega P(x,u)\,dx \le B_0, \quad u \in S_{\ell,2}. \tag{11.80}$$

Moreover, I claim that

$$G(w) \to \infty \text{ as } \|w\| \to \infty, \quad w \in M_\ell. \tag{11.81}$$

Assuming this, we note that there is an $R > 0$ sufficiently large that

$$\sup_B G \le \inf_A G, \tag{11.82}$$

where $A = M_\ell \cap \partial B_R$ and $B = S_{\ell,2}$. This is not quite (3.4). To correct the situation, we let $G_1 = -G$. Then (11.82) becomes

$$\sup_A G_1 \le \inf_B G_1. \tag{11.83}$$

We know from Lemma 11.11 that A links B [hm]. Consequently, we may apply Theorem 3.4 to conclude that there is a sequence $\{u_k\} \subset D$ such that (11.65) holds with G replaced by G_1. Now c satisfies the estimates

$$\inf_B G_1 \le c \le \sup_{M\cap B_R} G_1 \tag{11.84}$$

or

$$\inf_{M\cap B_R} G \le -c \le \sup_B G. \tag{11.85}$$

By (11.14), (11.56), and (11.80), we see that c satisfies (11.66) as well. Thus, for the given sequence, (11.65), (11.67)–(11.69) hold with c replaced by $-c$, while c satisfies (11.66). If we assume that (11.70) holds, we can reason as in the proof of Theorem 11.1 that there is a renamed subsequence of $\{\tilde{u}_k\}$ converging in D to a function $\tilde{u} \in D$ and a.e. in Ω. Moreover, the function $\tilde{u}$ satisfies (11.72) and (11.73). Also, (11.74) holds with c replaced by $-c$. But then (11.15) and (11.16) imply

$$c = \lim \int_\Omega H(x,u_k)dx \ge \int_{\Omega_+} H_+(x)dx + \int_{\Omega_-} H_-(x)dx - \int_{\Omega_0} W_0(x)\,dx > B_1. \tag{11.86}$$

This contradiction shows that the ρ_k are bounded, and we can now employ standard techniques to obtain a solution of

$$G(u) = c_1, \quad -B_1 \le c_1 \le B_0, G'(u) = 0. \tag{11.87}$$

This produces the desired result.

It therefore remains only to prove (11.81). Let $\{w_k\} \subset M_\ell$ be a sequence such that $\rho_k = \|w_k\|_D \to \infty$. Let $\tilde{w}_k = w_k/\rho_k$. Then we have $\|\tilde{w}_k\|_D = 1$, and there is a renamed subsequence such that $\tilde{w}_k \to \tilde{w}$ weakly in D, strongly in $L^2(\Omega)$, and a.e. in Ω. Thus,

$$2\int_\Omega F(x, w_k)dx/\rho_k^2 \to a\|\tilde{w}^-\|^2 + b\|\tilde{w}^+\|^2. \tag{11.88}$$

Consequently,

$$\begin{aligned} G(w_k)/\rho_k^2 &\to 1 - a\|\tilde{w}^-\|^2 - b\|\tilde{w}^+\|^2 \\ &\geq 1 - \|\tilde{w}\|_D^2 + (\lambda_{\ell+1} - a)\|\tilde{w}^-\|^2 + (\lambda_{\ell+1} - b)\|\tilde{w}^+\|^2, \end{aligned}$$

which is positive since both a and b are less than $\lambda_{\ell+1}$. This completes the proof of the theorem. □

11.4 Notes and remarks

The presentation given here is from [124]. Further work was done in [102].

Chapter 12

Rotationally Invariant Solutions

12.1 Introduction

In this chapter (and the next) we study periodic solutions of the Dirichlet problem for the semilinear wave equation. In this chapter we study radially symmetric solutions for the problem

$$\Box u := u_{tt} - \Delta u = f(t, x, u), \quad t \in \mathbb{R}, \ x \in B_R, \tag{12.1}$$

$$u(t, x) = 0, \quad t \in \mathbb{R}, \ x \in \partial B_R, \tag{12.2}$$

$$u(t + T, x) = u(t, x), \ t \in \mathbb{R}, \quad x \in B_R, \tag{12.3}$$

where

$$B_R = \{x \in \mathbb{R}^n : |x| < R\}. \tag{12.4}$$

In this case we have

$$f(t, x, u) = f(t, |x|, u), \quad x \in B_R.$$

Our basic assumption is that the ratio R/T is rational. Thus, we can write

$$8R/T = a/b, \tag{12.5}$$

where a, b are relatively prime positive integers. We show that

$$n \not\equiv 3 \qquad (\mathrm{mod}(4, a)) \tag{12.6}$$

implies that the linear problem corresponding to (12.1)–(12.3) has no essential spectrum. If

$$n \equiv 3 \qquad (\mathrm{mod}(4, a)), \tag{12.7}$$

then the essential spectrum of the linear operator consists of precisely one point:

$$\lambda_0 = -(n - 3)(n - 1)/4R^2. \tag{12.8}$$

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This shows that the spectrum has at most one limit point. We can then consider the nonlinear case

$$f(t, r, s) = \mu s + p(t, r, s), \tag{12.9}$$

where μ is a point in the resolvent set, $r = |x|$, and

$$|p(t, r, s)| \le C(|s|^{\theta} + 1), \quad s \in \mathbb{R}, \tag{12.10}$$

for some number $\theta < 1$. Our main theorem is

Theorem 12.1. *If (12.6) holds, then (12.1)–(12.3) has a weak rationally invariant solution. If (12.7) holds and $\lambda_0 < \mu$, assume, in addition, that $p(t, r, s)$ is nondecreasing in s. If $\mu < \lambda_0$, assume that $p(t, r, s)$ is nonincreasing in s. Then (12.1)–(12.3) has a weak rotationally invariant solution.*

Our proof of this theorem will make use of Theorem 10.2. For the definition of essential spectrum, cf., e.g., [126] and [106].

12.2 The spectrum of the linear operator

In proving Theorem 12.1, we shall need to calculate the spectrum of the linear operator $\Box$ applied to periodic rotationally symmetric functions. Specifically, we shall need

Theorem 12.2. *Let L_0 be the operator*

$$L_0 u = u_{tt} - u_{rr} - r^{-1}(n-1)u_r \tag{12.11}$$

applied to functions $u(t, r)$ in $C^{\infty}(\bar{\Omega})$ satisfying

$$u(T, r) = u(0, r), \quad u_t(T, r) = u_t(0, r), \quad 0 \le r \le R, \tag{12.12}$$

$$u(t, R) = u_R(t, 0) = 0, \quad t \in \mathbb{R}, \tag{12.13}$$

where $\Omega = [0, T] \times [0, R]$. Then L_0 is symmetric on $L^2(\Omega, \rho)$, where $\rho = r^{n-1}$. Assume that $8R/T = a/b$, where a, b are relatively prime integers [i.e., $(a, b) = 1$]. Then L_0 has a self-adjoint extension L having no essential spectrum other than the point $\lambda_0 = -(n-3)(n-1)/4R^2$. If $n \not\equiv 3 \pmod{4, a}$, then L has no essential spectrum. If $n \equiv 3 \pmod{4, a}$, then the essential spectrum of L is precisely the point λ_0.

Proof. Let $\nu = (n-2)/2$ and let γ be a positive root of $J_\nu(x) = 0$, where J_ν is the Bessel function of the first kind. Set

$$\varphi(r) = J_\nu(\gamma r/R)/r^{\nu}. \tag{12.14}$$

Then

$$\varphi'' + (n-1)\varphi'/r = (x^2 J_\nu'' + x J_\nu' - \nu^2 J_\nu)/r^{\nu+2} = -\gamma^2 J_\nu/R^2. \tag{12.15}$$

If

$$\psi(t,r) = \varphi(r)e^{2\pi ikt/T}, \tag{12.16}$$

then

$$L_0\psi = [(\gamma/R)^2 - (2\pi k/T)^2]\psi. \tag{12.17}$$

Let γ_j be the jth positive root of $J_\nu(x) = 0$, and set

$$\psi_{jk}(t,r) = r^{-\nu} J_\nu(\gamma_j r/R)e^{2\pi ikt/T}. \tag{12.18}$$

Then $\psi_{jk}(t,r)$ is an eigenfunction of L_0 with eigenvalue

$$\lambda_{jk} = (\gamma_j/R)^2 - (2\pi k/T)^2. \tag{12.19}$$

It is easily checked that the functions ψ_{jk}, when normalized, form a complete orthonormal sequence in $L^2(\Omega,\rho)$. We shall show that the corresponding eigenvalues (12.19) are not dense in $\mathbb{R}$. It will then follow that L_0 has a self-adjoint extension L with spectrum equal to the closure of the set $\{\lambda_{jk}\}$ (cf., e.g., [126]). Now

$$\gamma_j = \beta_j - (\mu-1)/8\beta_j + O(\beta_j^{-3}) \text{ as } \beta_j \to \infty, \tag{12.20}$$

where

$$\beta_j = \pi\left(j + \frac{1}{2}\nu - \frac{1}{4}\right), \quad \mu = 4\nu^2 \tag{12.21}$$

(cf., e.g., [157]). Thus,

$$\begin{aligned} \lambda_{jk}R^2 &= [\beta_j - \tau_k - (\mu-1)/8\beta_j + O(\beta_j^{-3})] \\ &\quad \cdot[\beta_j + \tau_k - (\mu-1)/8\beta_j + O(\beta_j^{-3})] \\ &= \beta_j^2 - \tau_k^2 - (\mu-1)/4 + O(\beta_j^{-2}) \end{aligned}$$

where $\tau_k = 2k\pi R/T$. (We may assume $k \geq 0$.) Now

$$\beta_j - \tau_k = \pi\left(j + \frac{1}{2}\nu - \frac{1}{4} - ak/4b\right) = \pi[(4j+n-3)b - ak]/4b. \tag{12.22}$$

Since the expression in the brackets is an integer, we see that either $\beta_j = \tau_k$ or

$$|\beta_j - \tau_k| \geq \pi/4b. \tag{12.23}$$

Thus,

$$\lim_{\substack{j,|k|\to\infty \\ \beta_j=\tau_k}} \lambda_{jk} = -(\mu-1)/4R^2 = \lambda_0 \tag{12.24}$$

and

$$\lim_{\substack{j,|k|\to\infty \\ \beta_j \neq \tau_k}} |\lambda_{jk}| = \infty. \tag{12.25}$$

If $n-3$ is not a multiple of $(4, a)$, then

$$\beta_j - \tau_k = \pi((4j + n - 3) - ak/b)/4 \tag{12.26}$$

can never vanish. To see this, note that if $(b, k) \neq b$, then ak/b is not an integer. Hence, $\beta_j \neq \tau_k$. If $b = (b, k)$, then

$$(n-3) \neq ak' - 4j \quad \forall j,\ k' = k/b. \tag{12.27}$$

Thus, in this case we always have $\beta_j \neq \tau_k$ and $|\lambda_{jk}| \to \infty$ as $j, k \to \infty$. On the other hand, if $n \equiv 3(\mathrm{mod}(4, a))$, then there are an infinite number of positive integers j, k' such that

$$n - 3 = ak' - 4j. \tag{12.28}$$

Hence, the point λ_0 is a limit point of eigenvalues. Consequently, it is in $\sigma_e(L)$. This completes the proof. $\square$

12.3 The nonlinear case

We now turn to the problem of solving (12.1)–(12.3). If one is searching for rotationally invariant solutions, the problem reduces to

$$Lu = f(t, r, u), \quad u \in D(L) \tag{12.29}$$

where L is the self-adjoint extension of the operator L_0 given in Theorem 12.2. Under the hypotheses of that theorem, the spectrum of L is discrete. We assume that

$$f(t, r, s) = \mu s + p(t, r, s), \tag{12.30}$$

where μ is a point in the resolvent set of L and $p(t, r, s)$ is a Carathéodory function on $\Omega \times \mathbb{R}$ such that

$$|p(t, r, s)| \leq C(|s|^\theta + 1), \quad s \in \mathbb{R}, \tag{12.31}$$

for some number $\theta < 1$. We have

Theorem 12.3. *Let $f(t, r, s)$ satisfy (12.30) and (12.31), and assume the hypotheses of Theorem 12.2. If*

$$n \not\equiv 3 \quad (\mathrm{mod}(4, a)), \tag{12.32}$$

make no further assumptions. If

$$n \equiv 3 \quad (\mathrm{mod}(4, a)) \tag{12.33}$$

and $\lambda_0 < \mu$, assume that $p(t, r, s)$ is nondecreasing in s. If (12.33) holds and $\mu < \lambda_0$, assume that $p(t, r, s)$ is nonincreasing in s. Then (12.29) has at least one weak solution.

Proof. Since μ is in the resolvent set of L, there is a $\delta > 0$ such that

$$|\lambda_{jk} - \mu| \geq \delta \quad \forall\, j, k, \tag{12.34}$$

where the λ_{jk} are given by (12.19). Each $u \in L^2(\Omega, \rho)$ can be expanded in the form

$$u = \sum \alpha_{jk}\psi_{jk}(t, r), \tag{12.35}$$

where the ψ_{jk} are given by (12.18). Let N_0 be the subspace of those $u \in L^2(\Omega, \rho)$ for which $\alpha_{jk} = 0$ if $\beta_j \neq \tau_k$ (cf. the proof of Theorem 12.2). For $u \in N_0$,

$$u = \sum_{[0]} \alpha_{jk}\psi_{jk}(t, r), \tag{12.36}$$

where summation is taken over those j, k for which $\beta_j = \tau_k$. Let E be the subspace of $L^2(\Omega, \rho)$ consisting of those u for which

$$\|u\|_E^2 = \sum |\lambda_{jk} - \mu||\alpha_{jk}|^2 \tag{12.37}$$

is finite. With this norm, E becomes a separable Hilbert space. Note that $E \subset D(|L|^{1/2})$ and the embedding of $E \ominus N_0$ into $L^2(\Omega, \rho)$ is compact [we use (12.25) for this purpose]. Let

$$G(u) = ([L - \mu]u, u) - 2\int\int_\Omega P(t, r, u)\rho\, dt\, dr, \quad u \in E, \tag{12.38}$$

where

$$P(t, r, s) = \int_0^s p(t, r, \sigma)\, d\sigma, \tag{12.39}$$

and then scalar product is that of $L^2(\Omega, \rho)$. One checks readily that G is a C^1-functional on E with

$$(G'(u), v)/2 = ([L - \mu]u, v) - (p(u), v), \quad u, v \in E, \tag{12.40}$$

where we write $p(u)$ in place of $p(t, r, u)$. This shows that u is a weak solution of (12.29) iff $G'(u) = 0$. Let N be the subspace of E spanned by the ψ_{jk} corresponding to those $\lambda_{jk} < \mu$ and let M denote the subspace of E spanned by the rest. Thus, $M = N^\perp$ in E. Assume first that $N \cap N_0 = \{0\}$. Then

$$\|u\|_E^2 = \sum (\mu - \lambda_{jk})|\alpha_{jk}|^2, \quad u \in N. \tag{12.41}$$

Thus,

$$\begin{aligned} G(v) &= -\|v\|_E^2 - 2\int\int_\Omega P(t, r, v)\rho\, dt\, dr \\ &\leq -\|v\|_E^2 + C\int\int_\Omega (|v|^{1+\theta} + |v|)\rho\, dt\, dr \\ &\leq -\|v\|_E^2 + C'(\|v\|^{1+\theta} + \|v\|) \to -\infty, \quad \|v\|_E \to \infty,\ v \in N. \end{aligned}$$

If $w \in M$,

$$G(w) \geq \delta\|w\|^2 - C(\|w\|^{1+\theta} + \|w\|) \geq -K, \quad w \in M. \tag{12.42}$$

We can now make use of Theorem 10.2. If Q is a large ball in N, then

$$\sup_{\partial Q} G \leq \inf_M G. \tag{12.43}$$

By Example 1 of Section 10.3, ∂Q links M weakly. Moreover, if $\{u_k\} \subset E$ is a sequence such that $v_k = Pu_k \to v = Pu$ weakly on N and $w_k = (I - P)u_k \to w = (I - P)u$ strongly in M, where P is the projection of E onto N, then $\{u_k\}$ has a renamed subsequence that converges strongly in $L^2(\Omega, \rho)$. The reason is that $\{v_k\}$ has such a subsequence because the embedding of $E \ominus N_0$ in $L^2(\Omega, \rho)$ is compact. Thus, $G'(u_n) \to G'(u)$ weakly in E. Hence, all of the hypotheses of Theorem 10.2 are satisfied, and we can conclude that there is a sequence $\{u_k\}$ satisfying (10.3). Write $u_k = v_k + w_k + y_k$, where $v_k \in N,\ w_k \in M \ominus N_0,\ y_k \in N_0$. Then

$$(G'(u_k), v_k)/2 = ([L - \mu]u_k, v_k) - (p(u_k), v_k) \tag{12.44}$$

and, consequently,

$$\|v_k\|_E^2 \leq \|G'(u_k)\|\|v_k\|_E/2 + C\|v_k\|(\|u_k\|^\theta + 1) \tag{12.45}$$

in view of (12.31) and (12.37). Similarly,

$$\|w_k\|_E^2 \leq \|G'(u_k)\|\|w_k\|_E/2 + C\|w_k\|(\|u_k\|^\theta + 1). \tag{12.46}$$

If $N_0 = \{0\}$, then it follows from (12.45) and (12.46) that $\|u_k\|_E$ is bounded and consequently there is a renamed subsequence that converges weakly in E and strongly in $L^2(\Omega, \rho)$ to a function u. Thus, $G'(u_k) \to G'(u)$ weakly. But $G'(u_k) \to 0$. Consequently $G'(u) = 0$ and the proof for this case is complete. If $N_0 \neq \{0\}$, we note that

$$\|y_k\|_E^2 \leq \|G'(u_k)\|\|y_k)\|_E/2 + C\|y_k||(\|u_k\|^\theta + 1) \tag{12.47}$$

as well. Again, this together with (12.45) and (12.46) implies that $\|u_k||_E$ is bounded and has a renamed subsequence that converges weakly in E and such that $u'_k = v_k + w_k$ converges strongly in $L^2(\Omega, \rho)$. Now

$$\begin{aligned}(G'(u_k), y_k - y)/2 \quad = \quad & ([L - \mu](y_k - y), y_k - y) \\ & -(p(u_k) - p(u'_k + y), y_k - y) \\ & +(p(u'_k + y) - p(u), y_k - y) \\ & +([L - \mu]y, y_k - y),\end{aligned} \tag{12.48}$$

where $y_k \to y$ weakly in E and $L^2(\Omega, \rho)$ and $u'_k \to u'$ weakly in E and strongly in $L^2(\Omega, \rho)$. By hypothesis

$$(p(u_k) - p(u'_k + y), y_k - y) \geq 0 \tag{12.49}$$

if $\mu > \lambda_0$. Moreover,

$$\begin{aligned}(G'(u_k), y_k - y) &\to 0,\\ (p(u'_k + y) - p(u), y_k - y) &\to 0,\end{aligned}$$

and

$$([L-\mu]y, y_k - y) \to 0. \tag{12.50}$$

Hence

$$\|y_k - y\|_E^2 \leq o(1), \quad k \to \infty. \tag{12.51}$$

This shows that $y_k \to y$ in E, and the proof proceeds as before. If $\lambda_0 > \mu$, we apply Theorem 12.1 to $-G(u)$ and come to the same conclusion. In this case the inequality in (12.49) is reversed. This completes the proof. □

12.4 Notes and remarks

Many authors have studied the one-dimensional periodic-Dirichlet problem for the semilinear wave equation

$$\begin{gathered}u_{tt} - u_{xx} = p(t,x,u), \quad t \in \mathbb{R},\ x \in (0,\pi),\\ u(t,x) = 0, \quad t \in \mathbb{R},\ x = 0,\ x = \pi,\\ u(t+2\pi, x) = u(t,x), \quad t \in \mathbb{R},\ x \in (0,\pi)\end{gathered}$$

A basic problem in this one dimensional case is that the null space N of the linear part $\Box u = u_{tt} - u_{xx}$ is infinite dimensional. On the other hand, $\Box$ has a compact inverse on the orthogonal complement of N. In contrast to this, the higher dimensional periodic-Dirichlet problem for the semilinear wave equation (13.1)–(13.2) has the additional difficulty that $\Box$ does not have a compact inverse on $N^\perp$. In fact, it has a sequence of eigenvalues of infinite multiplicities stretching from $-\infty$ to ∞. This is a serious complication that causes all of the methods used to solve the one-dimensional case to fail.

Recently some authors have examined the radially symmetric counterpart of (13.1)–(13.3) that was considered in this chapter (cf. [146], [15],[14],[23],and [121]). It is assumed that the function $f(t,x,u)$ is radially symmetric in x. This allows one to reduce the problem to

$$\begin{aligned}&u_{tt} - u_{rr} - r^{-1}(n-1)u_r = f(t,r,u),\\ &\qquad u(2\pi, r) = u(0,r), \quad u_t(2\pi, r) = u_t(0,r), \quad 0 \leq r \leq R,\\ &\qquad u(t,R) = u_R(t,R) = 0, \quad t \in \mathbb{R}.\end{aligned}$$

This is much more difficult than the one-dimensional problem for the wave equation, but the techniques used in solving it cannot be used to solve the n-dimensional problem for the wave operator when the region and functions are not radially symmetric. This will be addressed in Chapter 13.

Chapter 13

Semilinear Wave Equations

13.1 Introduction

In this chapter we shall consider the higher-dimensional periodic-Dirichlet problem for the semilinear wave equation

(13.1) $$\Box u \equiv u_{tt} - \Delta u = p(x,t,u), \quad (x,t) \in \Omega,$$

(13.2) $$u(x,t) = 0, \quad t \in \mathbb{R},\ x \in \partial(0,\pi)^n,$$

(13.3) $$u(x,t+2\pi) = u(x,t), \quad (x,t) \in \Omega,$$

where $\Omega = (0,\pi)^n \times (0,2\pi)$. Here, $x = (x_1,\dots,x_n) \in \mathbb{R}^n$ and

$$(0,\pi)^n = \{x \in \mathbb{R}^n : 0 < x_k < \pi,\ 1 \le k \le n\}.$$

In studying this problem, we shall make use of the theory of saddle points.

13.2 Convexity and lower semi-continuity

A set M is called **convex** if $(1-t)w_0 + tw_1 \in M$ whenever $w_0, w_1 \in M$ and $0 \le t \le 1$.

Let M be a convex subset of a Hilbert space E, and let G be a functional (real-valued function) defined on M. We call G **convex** on M if

$$G((1-t)w_0 + tw_1) \le (1-t)G(w_0) + tG(w_1), \quad w_0, w_1 \in M,\ 0 \le t \le 1.$$

We call it **strictly convex** if the inequality is strict when $0 < t < 1,\ w_0 \ne w_1$.

$G(v)$ is called **upper semi-continuous** (u.s.c.) at $w_0 \in M$ if $w_k \to w_0 \in M$ implies

$$G(w_0) \ge \limsup G(w_k).$$

It is called **lower semi-continuous** (l.s.c.) if the inequality is reversed and lim sup is replaced by lim inf. We have

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Lemma 13.1. *If M is closed, convex, and bounded in E and G is convex and l.s.c. on M, then there is a point $w_0 \in M$ such that*

$$G(w_0) = \min_M G. \tag{13.4}$$

If G is strictly convex, then w_0 is unique.

In proving Lemma 13.1, we shall make use of

Lemma 13.2. *If $u_k \rightharpoonup u$ in E, then there is a renamed subsequence such that $\bar{u}_k \to u$, where*

$$\bar{u}_k = (u_1 + \cdots + u_k)/k. \tag{13.5}$$

Proof. We may assume that $u = 0$. Take $n_1 = 1$, and inductively pick $n_2, n_3, \ldots,$ so that

$$|(u_{n_k}, u_{n_1})| \leq \frac{1}{k}, \ldots, \quad |(u_{n_k}, u_{n_{k-1}})| \leq \frac{1}{k}.$$

This can be done since

$$(u_n, u_{n_j}) \to 0 \text{ as } n \to \infty, \quad 1 \leq j \leq k.$$

Since

$$\|u_k\| \leq C$$

for some C, we have

$$\begin{aligned} \|\bar{u}_k\|^2 &= \left[\sum_{j=1}^{k} \|u_j\|^2 + 2\sum_{j=1}^{k}\sum_{i=1}^{j-1}(u_i, u_j)\right] / k^2 \\ &\leq \left[kC^2 + 2\sum_{j=1}^{k}\sum_{i=1}^{j}\frac{1}{j}\right] / k^2 \\ &\leq (C^2 + 2)/k \to 0. \end{aligned}$$

□

Lemma 13.3. *If $G(u)$ is convex and l.s.c. on E, and $u_k \rightharpoonup u$, then*

$$G(u) \leq \liminf G(u_k).$$

Proof. Let

$$L = \liminf G(u_k).$$

Then there is a renamed subsequence such that $G(u_k) \to L$. Let $\varepsilon > 0$ be given. Then

$$L - \varepsilon < G(u_k) < L + \varepsilon \tag{13.6}$$

for all but a finite number of k. Remove a finite number and rename the subsequence so that (13.6) holds for all k. Moreover, there is a renamed subsequence such that $\bar{u}_k \to u$ by Lemma 13.2, where $\bar{u}_k$ is given by (13.5). Thus,

$$\begin{aligned} G(u) &\leq \liminf G(\bar{u}_k) = \liminf G\left(\frac{1}{k}\sum_{j=1}^{k} u_j\right) \\ &\leq \liminf \frac{1}{k}\sum_{j=1}^{k} G(u_j) \leq \liminf \frac{1}{k}\cdot k(L+\varepsilon) = L+\varepsilon. \end{aligned}$$

Since ε was arbitrary, we see that $G(u) \leq L$, and the proof is complete. □

A subset $M \subset E$ is called **weakly closed** if $u \in M$ whenever there is a sequence $\{u_k\} \subset M$ converging weakly to u in E. A weakly closed set is closed in a stronger sense than an ordinary closed set. It follows from Lemma 13.2 that

Lemma 13.4. *If M is a closed, convex subset of E, then it is weakly closed in E.*

Proof. Suppose $\{u_k\} \subset M$ and $u_k \rightharpoonup u$ in E. Then, by Lemma 13.2, there is a renamed subsequence such that $\bar{u}_k \to u$, where $\bar{u}_k$ is given by (13.5). Since M is convex, each $\bar{u}_k$ is in M. Since M is closed, we see that $u \in M$. □

We can now give the proof of Lemma 13.1.

Proof. Let

$$\alpha = \inf_M G.$$

(At this point we do not know if $\alpha \neq -\infty$.) Let $\{w_k\} \subset M$ be a sequence such that $G(w_k) \to \alpha$. Since M is bounded, we see that there is a renamed subsequence such that $w_k \rightharpoonup w_0$. Since M is closed and convex, it is weakly closed (Lemma 13.4). Hence, $w_0 \in M$. By Lemma 13.3, $G(w_0) \leq \liminf G(w_k) = \alpha$. Since $G(w_0) \geq \alpha$, we see that (13.4) holds. So far, we have only used the convexity of G. We use the strict convexity to show that w_0 is unique. If there were another element $w_1 \in M$ such that $G(w_1) = \alpha$, then we would have

$$G\left(\frac{1}{2}w_0 + \frac{1}{2}w_1\right) < \frac{1}{2}[G(w_0) + G(w_1)] = \alpha,$$

which is impossible from the definition of α. This completes the proof. □

We also have

Lemma 13.5. *If M is closed and convex, G is convex, is l.s.c., and satisfies*

$$G(u) \to \infty \text{ as } \|u\| \to \infty, \quad u \in M \tag{13.7}$$

(if M is unbounded), then G is bounded from below on M and has a minimum there. If G is strictly convex, this minimum is unique.

Proof. If M is bounded, then Lemma 13.5 follows from Lemma 13.1. Otherwise, let u_0 be any element in M. By (13.7), there is an $R \geq \|u_0\|$ such that

$$G(u) \geq G(u_0), \quad u \in M, \ \|u\| \geq R.$$

By Lemma 13.1, G is bounded from below on the set

$$M_R = \{w \in M : \|w\| \leq R\}$$

and has a minimum there. A minimum of G on M_R is a minimum of G on M. Hence, G is bounded from below on M and has a minimum there. □

13.3 Existence of saddle points

We say that (v_0, w_0) is a **saddle point** of G if

$$G(v, w_0) \leq G(v_0, w_0) \leq G(v_0, w), \quad v \in N, w \in M. \tag{13.8}$$

We now present some sufficient conditions for the existence of saddle points. Let M, N be closed, convex subsets of a Hilbert space, and let $G(v, w) : M \times N \to \mathbb{R}$ be a functional such that $G(v, w)$ is convex and l.s.c. in w for each $v \in N$, and concave and u.s.c. in v for each $w \in M$. If M is unbounded, assume also that there is a $v_0 \in N$ such that

$$G(v_0, w) \to \infty \ \text{ as } \ \|w\| \to \infty, \quad w \in M. \tag{13.9}$$

If N is unbounded, assume that there is a $w_0 \in M$ such that

$$G(v, w_0) \to -\infty \ \text{ as } \ \|v\| \to \infty, \quad v \in N. \tag{13.10}$$

[If M is bounded, then (13.9) is automatically satisfied; the same is true for (13.10) when N is bounded.] We have

Theorem 13.6. *Under the above hypotheses, G has at least one saddle point.*

Proof. Assume first that M, N are bounded. Then, for each $v \in N$, there is at least one point w where $G(v, w)$ achieves its minimum (Lemma 13.1). Let

$$J(v) = \min_{w \in M} G(v, w).$$

Since $J(v)$ is the minimum of a family of functionals that are concave and u.s.c., it is also concave and u.s.c. In fact, if

$$v_t = (1 - t)v_0 + tv_1, \quad t \in [0, 1],$$

then

$$G(v_t, w) \geq (1 - t) \min_{\hat{w} \in M} G(v_0, \hat{w}) + t \min_{\hat{w} \in M} G(v_1, \hat{w}), \quad w \in M.$$

Since this is true for each $w \in M$, we see that

$$J(v_t) \geq (1-t)J(v_0) + tJ(v_1). \tag{13.11}$$

Similarly, if $v_k \to v \in N$, then we have

$$J(v_k) \leq G(v_k, w), \quad w \in M.$$

Thus,

$$\limsup J(v_k) \leq \limsup G(v_k, w) \leq G(v, w), \quad w \in M.$$

Since this is true for each $w \in M$, we have

$$\limsup J(v_k) \leq \inf_{w \in M} G(v, w) = J(v). \tag{13.12}$$

Therefore, $J(v)$ is concave and u.s.c. Consequently, $J(v)$ has a maximum point $\bar{v}$ satisfying

$$J(v) \leq J(\bar{v}), \quad v \in N$$

(Lemma 13.1). In particular, we have

$$J(\bar{v}) = \min_{\hat{w} \in M} G(\bar{v}, \hat{w}) \leq G(\bar{v}, w), \quad w \in M. \tag{13.13}$$

Let v be an arbitrary point in N, and let

$$v_\theta = (1-\theta)\bar{v} + \theta v, \quad 0 \leq \theta \leq 1.$$

Since G is concave in v, we have

$$G(v_\theta, w) \geq (1-\theta)G(\bar{v}, w) + \theta G(v, w).$$

Consequently,

$$\begin{aligned} J(\bar{v}) &\geq J(v_\theta) \\ &= G(v_\theta, w_\theta) \\ &\geq (1-\theta)G(\bar{v}, w_\theta) + \theta G(v, w_\theta) \\ &\geq (1-\theta)J(\bar{v}) + \theta G(v, w_\theta), \end{aligned}$$

where w_θ is any point in M such that

$$G(v_\theta, w_\theta) = \min_{w \in M} G(v_\theta, w).$$

This gives

$$J(\bar{v}) \geq G(v, w_\theta), \quad v \in N, \ 0 < \theta \leq 1. \tag{13.14}$$

Let $\{\theta_k\}$ be a sequence converging to 0, and let $v_k = v_{\theta_k}$, $w_k = w_{\theta_k}$. Then $v_k \to \bar{v}$. Since M is bounded, there is a renamed subsequence such that $w_k \rightharpoonup \bar{w}$. Since

$$(1-\theta)G(\bar{v}, w_\theta) + \theta G(v, w_\theta) \leq G(v_\theta, w_\theta) \leq G(v_\theta, w), \quad w \in M,$$

we have

$$(1-\theta_k)G(\bar{v}, w_k) + \theta_k J(v) \le G(v_k, w), \quad w \in M.$$

In the limit this gives

$$G(\bar{v}, \bar{w}) \le G(\bar{v}, w), \quad w \in M$$

(cf. Lemma 13.3). Since

$$J(\bar{v}) \ge G(v, w_k),$$

we have

$$G(v, \bar{w}) \le J(\bar{v}) \le G(\bar{v}, w), \quad v \in N, \ w \in M,$$

in view of (13.13) and (13.14). Take $v = \bar{v}$ and $w = \bar{w}$. Then

$$G(\bar{v}, \bar{w}) \le J(\bar{v}) \le G(\bar{v}, \bar{w}),$$

showing that

$$G(\bar{v}, \bar{w}) = J(\bar{v})$$

and

$$G(v, \bar{w}) \le G(\bar{v}, \bar{w}) \le G(\bar{v}, w), \quad v \in N, \ w \in M.$$

Thus, $(\bar{v}, \bar{w})$ is a saddle point.

Now, we remove the restriction that M, N are bounded. Let R be so large that $\|v_0\| < R, \ \|w_0\| < R$. The sets

$$M_R = \{w \in M : \|w\| \le R\}, \quad N_R = \{v \in N : \|v\| \le R\}$$

are closed, convex, and bounded. By what we have already proved, there is a saddle point $(\bar{v}_R, \bar{w}_R)$ such that

$$G(v, \bar{w}_R) \le G(\bar{v}_R, \bar{w}_R) \le G\bar{v}_R, w), \quad v \in N_R, \ w \in M_R. \tag{13.15}$$

In particular, we have

$$G(v_0, \bar{w}_R) \le G(\bar{v}_R, \bar{w}_R) \le G(\bar{v}_R, w_0).$$

Since $G(v_0, w)$ is convex, is l.s.c., and satisfies (13.9), it is bounded from below on M (Lemma 13.5). Thus,

$$G(v_0, \bar{w}_R) \ge A > -\infty.$$

Similarly, $G(v, w_0)$ is bounded from above. Hence,

$$G(\bar{v}_R, w_0) \le B < \infty.$$

Combining these with (13.15), we have

$$A \le G(v_0, \bar{w}_R) \le G(\bar{v}_R, \bar{w}_R) \le G(\bar{v}_R, w_0) \le B.$$

By (13.9) and (13.10), the sequences $\{\bar{v}_R\}, \ \{\bar{w}_R\}$ are bounded. Hence, there are renamed subsequences such that

$$\bar{v}_R \rightharpoonup \bar{v}, \ \bar{w}_R \rightharpoonup \bar{w} \ \text{ as } \ R \to \infty$$

and

$$G(\bar{v}_R, \bar{w}_R) \to \lambda \quad \text{as} \quad R \to \infty.$$

In view of (13.15), we have in the limit

$$G(v, \bar{w}) \le \lambda \le G(\bar{v}, w), \qquad v \in N, \ w \in M.$$

This shows that $\lambda = G(\bar{v}, \bar{w})$, and $(\bar{v}, \bar{w})$ is a saddle point. The theorem is completely proved. $\square$

13.4 Criteria for convexity

If G is a differentiable functional on a Hilbert space E, there are simple criteria that can be used to verify the convexity of G. We have

Theorem 13.7. *Let G be a differentiable functional on a closed, convex subset M of E. Then G is convex on E iff it satisfies any of the following inequalities for $u_0, u_1 \in M$:*

$$(G'(u_0), u_1 - u_0) \le G(u_1) - G(u_0) \tag{13.16}$$

$$(G'(u_1), u_1 - u_0) \ge G(u_1) - G(u_0), \tag{13.17}$$

$$(G'(u_1) - G'(u_0), u_1 - u_0) \ge 0. \tag{13.18}$$

Moreover, it will be strictly convex iff there is strict inequality in any of them when $u_0 \neq u_1$.

Proof. Let $u_t = (1-t)u_0 + tu_1, \ 0 \le t \le 1$, and $\varphi(t) = G(u_t)$. If G is convex, then

$$G(u_t) \le (1-t)G(u_0) + tG(u_1) \tag{13.19}$$

or

$$\varphi(t) \le (1-t)\varphi(0) + t\varphi(1), \qquad 0 \le t \le 1. \tag{13.20}$$

In particular, the slope of φ at $t = 0$ is less than or equal to the slope of the straight line connecting $(0, \varphi(0))$ and $(1, \varphi(1))$. Thus, $\varphi'(0) \le \varphi(1) - \varphi(0)$, and this is merely (13.16). Reversing the roles of u_0, u_1 produces (13.17). We obtain (13.18) by subtracting (13.16) from (13.17). Conversely, (13.18) implies

$$\begin{aligned} \varphi'(t) - \varphi'(s) &= (G'(u_t) - G'(u_s), u_1 - u_0) \\ &= (G'(u_t) - G'(u_s), u_t - u_s)/(t-s) \ge 0, \qquad 0 \le s < t \le 1. \end{aligned}$$

Thus,

$$\varphi'(t) \ge \varphi'(s), \qquad 0 \le s \le t \le 1,$$

which implies (13.20). Since this is equivalent to (13.19), we see that G is convex. If G is strictly convex, we obtain strict inequalities in (13.16)–(13.18), and strict inequalities in any of them implies strict inequalities in (13.20) and (13.19). $\square$

Corollary 13.8. *Let G be a differentiable functional on a closed, convex subset M of E. Then G is concave on E iff it satisfies any of the following inequalities for $u_0, u_1 \in M$:*

$$(G'(u_0), u_1 - u_0) \geq G(u_1) - G(u_0), \tag{13.21}$$

$$(G'(u_1), u_1 - u_0) \leq G(u_1) - G(u_0), \tag{13.22}$$

$$(G'(u_1) - G'(u_0), u_1 - u_0) \leq 0. \tag{13.23}$$

Moreover, it will be strictly concave iff there is strict inequality in any of them when $u_0 \neq u_1$.

Proof. Note that $G(u)$ is concave iff $-G(u)$ is convex. □

13.5 Partial derivatives

Let M, N be closed subspaces of a Hilbert space H satisfying $H = M \oplus N$. Let $G(u)$ be a functional on H. We can consider "partial" derivatives of G in the same way we considered total derivatives. We keep $w = w_0 \in M$ fixed and consider $G(u)$ as a functional on N, where $u = v + w_0,\ v \in N$. If the derivative of this functional exists at $v = v_0 \in N$, we call it the **partial derivative** of G at $u_0 = v_0 + w_0$ with respect to $v \in N$ and denote it by $G'_N(u_0)$. Similarly, we can define the partial derivative $G'_M(u_0)$. We have

Lemma 13.9. *If G' exists at $u_0 = v_0 + w_0$, then $G'_M(u_0)$ and $G'_N(u_0)$ exist and satisfy*

$$(G'(u_0), u) = (G'_M(u_0), w) + (G'_N(u_0), v), \quad v \in N,\ w \in M. \tag{13.24}$$

Proof. By definition,

$$G(u_0 + u) = G(u_0) + (G'(u_0), u) + o(\|u\|), \quad u \in H.$$

Therefore,

$$G(u_0 + v) = G(u_0) + (G'(u_0), v) + o(\|v\|), \quad v \in N$$

and

$$G(u_0 + w) = G(u_0) + (G'(u_0), w) + o(\|w\|), \quad w \in M.$$

But

$$G(u_0 + v) = G(u_0) + (G'_N(u_0), v) + o(\|v\|), \quad v \in N$$

and

$$G(u_0 + w) = G(u_0) + (G'_M(u_0), w) + o(\|w\|), \quad w \in M.$$

In particular, we have

$$(G'(u_0) - G'_N(u_0), v) = o(\|v\|) \text{ as } \|v\| \to 0, \quad v \in N.$$

Thus,

$$(G'(u_0) - G'_N(u_0), tv) = o(|t|) \text{ as } |t| \to 0$$

for each fixed $v \in N$. This means that

$$(G'(u_0) - G'_N(u_0), v) = \frac{o(|t|)}{t} \to 0 \text{ as } t \to 0.$$

Hence,

$$(G'(u_0), v) = (G'_N(u_0), v), \quad v \in N.$$

Similarly,

$$(G'(u_0), w) = (G'_M(u_0), w), \quad w \in M.$$

These two identities combine to give (13.24). □

Lemma 13.10. *Under the hypotheses of Lemma 13.9, assume that G is differentiable on H, convex on M, and concave on N. Then,*

$$G(u) - G(u_0) \le (G'_N(u_0), v - v_0) + (G'_M(u), w - w_0), \tag{13.25}$$
$$u = v + w,\ u_0 = v_0 + w_0,\ v, v_0 \in N,\ w, w_0 \in M.$$

If G is either strictly convex on M or strictly concave on N (or both), then one has strict inequality in (13.25) when $u \ne u_0$.

Proof. This follows from Theorem 13.7 and its corollary. In fact, we have

$$\begin{aligned} G(u) - G(u_0) &= G(u) - G(v + w_0) + G(v + w_0) - G(u_0) \\ &\le (G'(u), w - w_0) + (G'(u_0), v - v_0). \end{aligned}$$

Apply Lemma 13.9. □

We also have

Lemma 13.11. *Under the hypotheses of Lemma 13.9, if $G'(u_0)$ exists and $u_0 = v_0 + w_0$ is a saddle point, then*

$$G'(u_0) = G'_M(u_0) = G'_N(u_0) = 0.$$

Proof. By definition,

$$G(v + w_0) \le G(u_0) \le G(v_0 + w), \quad v \in N,\ w \in M.$$

Since v_0 is a maximum point on N, we see that $G'_N(u_0) = 0$. Since w_0 is a minimum point on M, we have $G'_M(u_0) = 0$ for the same reason. We then apply Lemma 13.9. □

Corollary 13.12. *Under the hypotheses of Lemma 13.10, if G is either strictly convex on M or strictly concave on N (or both), then G has exactly one saddle point.*

Proof. This follows from inequality (13.25). □

13.6 The theorems

In solving problem (13.1)–(13.3) we take $p(x,t,\xi)$ to be a Carathéodory function on $\Omega \times \mathbb{R}$ which is 2π-periodic in t and satisfies

$$\theta_-\xi^2 \le \xi[p(x,t,\xi_1) - p(x,t,\xi_0)] \le \theta_+\xi^2, \quad \xi = \xi_1 - \xi_2 \in \mathbb{R}. \tag{13.26}$$

Let

$$\mu = (\mu_1, \ldots, \mu_n), \quad \mu_j \ge 0, \quad \mu_j \in \mathbb{Z}, \quad \mu^2 = |\mu|^2 = \sum \mu_j^2.$$

Let $\sigma(\Box)$ be the set of integers of the form $\lambda = \mu^2 - k^2$, and let $\rho(\Box) = \mathbb{R}\backslash\sigma(\Box)$. We shall prove

Theorem 13.13. *If*

$$[\theta_-, \theta_+] \subset \rho(\Box), \tag{13.27}$$

then there is a unique weak solution of (13.1)–(13.3).

Theorem 13.14. *If*

$$(\theta_-, \theta_+) \subset \rho(\Box), \tag{13.28}$$

and there are constants $\beta_\pm$ such that $\theta_- \le \beta_- \le \beta_+ \le \theta_+$, $[\beta_-, \beta_+] \subset \rho(\Box)$ and

$$\beta_- \le \liminf_{|\xi|\to\infty} 2P(x,t,\xi)/\xi^2 \le \limsup_{|\xi|\to\infty} 2P(x,t,\xi)/\xi^2 \le \beta_+ \tag{13.29}$$

uniformly in Ω, where

$$P(x,t,\xi) = \int_0^\xi p(x,t,s)ds,$$

then (13.1)–(13.3) has at least one weak solution.

Theorems 13.13 and 13.14 are proved in the next section.

13.7 The proofs

In this section we present the proof of Theorems 13.13 and 13.14. Let $x = (x_1, \ldots, x_n)$,

$$\varphi_\mu(x) = \sin \mu_1\xi_1 \cdots \sin \mu_n x_n/(\pi/2)^{n/2}.$$

Note that

$$(\varphi_\mu, \varphi_\nu) = \delta_{\mu\nu},$$

where the scalar product is that of $L^2([0,\pi]^n)$. We take

$$e_k(t) = e^{ikt}/(2\pi)^{1/2}, \quad t \in [0, 2\pi].$$

Then

$$(e_k, e_\ell) = \delta_{k\ell},$$

where the scalar product is that of $L^2([0, 2\pi])$. If

$$u = \sum \alpha_{\mu k} \varphi_\mu(x) e_k(t), \tag{13.30}$$

then

$$\Box u \equiv u_{tt} - \Delta u = \sum (\mu^2 - k^2) \alpha_{\mu k} \varphi_\mu(x) e_k(t).$$

Let γ be a fixed number in (θ_-, θ_+) and define

$$G(u) = \|\nabla u\|^2 - \|u_t\|^2 - 2 \int_\Omega P(x, t, u), \quad u \in E,$$

where the norm is that of $L^2(\Omega)$, and E is the set of all $u \in L^2(\Omega)$ of the form (13.30) such that

$$\|u\|_E^2 = \sum |\mu^2 - k^2 - \gamma| \cdot |\alpha_{\mu k}|^2 < \infty.$$

It is easily checked that $G \in C^1(E, \mathbb{R})$ and

$$(G'(u), v)/2 = (\Box u, v) - (p(u), v), \quad u, v \in E,$$

where we write $p(u)$ in place of $p(x, t, u)$. Let m_+ be the smallest point of $\sigma(\Box)$ above γ and m_- the largest point of $\sigma(\Box)$ below γ.

Let

$$M = \{u \in E : \alpha_{\mu k} = 0 \text{ when } \mu^2 - k^2 < \gamma\},$$

$$N = \{u \in E : \alpha_{\mu k} = 0 \text{ when } \mu^2 - k^2 > \gamma\}.$$

We have

Lemma 13.15.

$$(G'(v + w_1) - G'(v + w_0), w)/2 \geq \left(1 - \frac{\theta_+ - \gamma}{m_+ - \gamma}\right) \|w\|_E^2,$$

where $w = w_1 - w_0 \in M, \ v \in N.$

Proof. The left-hand side equals

$$\begin{aligned}(\Box w, w) &- (p(v + w_1) - p(v + w_0), w) \\ &\geq \|w\|_E^2 - (\theta_+ - \gamma)\|w\|^2 \\ &\geq \left(1 - \frac{\theta_+ - \gamma}{m_+ - \gamma}\right) \|w\|_E^2.\end{aligned}$$

$\square$

Lemma 13.16.

$$(G'(v_1 + w) - G'(v_0 + w), v)/2 \leq -\left(1 - \frac{\gamma - \theta_-}{\gamma - m_-}\right) \|v\|_E^2,$$

where $v = v_1 - v_0 \in N, \ w \in M.$

Proof. Now the left-hand side equals

$$\begin{aligned}(\Box v, v) - (p(v_1 + w) - p(v_0 + w), v) \\ &\leq -\|v\|_E^2 + (\gamma - \theta_-)\|v\|^2 \\ &\leq -\left(1 - \frac{\gamma - \theta_-}{\gamma - m_-}\right)\|v\|_E^2.\end{aligned}$$

□

We can now give the proof of Theorem 13.14.

Proof. By (13.28) and Lemmas 13.15 and 13.16, $G(v+w)$ is convex in w and concave in v. Moreover,

(13.31) $$G(w) \to \infty \text{ as } \|w\|_E \to \infty, \quad w \in M,$$

and

(13.32) $$G(v) \to -\infty \text{ as } \|v\|_E \to \infty, \quad v \in N.$$

To see this, let

$$w_j = \sum \alpha_{\mu k}^{(j)} \varphi_\mu e_k$$

be a sequence in M such that $\rho_j = \|w_j\|_E \to \infty$. Take $\tilde{w}_j = w_j/\rho_j$, and let $\varepsilon > 0$ be such that $\beta_+ < m_+ - \varepsilon$. Then there is a constant K such that

$$2P(x, t, \xi)/\xi^2 < \beta_+ + \varepsilon, \quad |\xi| > K.$$

Hence,

$$\begin{aligned}2\int_\Omega P(x, t, w_j)/\rho_j^2 &\leq \int_{|w_j|>K} (\beta_+ + \varepsilon)\tilde{w}_j^2 + \int_{|w_j|\leq K} P(x, t, w_j)/\rho_j^2 \\ &\leq \frac{\beta_+ + \varepsilon}{m_+ - \gamma} + C/\rho_j^2\end{aligned}$$

since $\|\tilde{w}_j\|_E = 1$. Therefore,

$$G(w_j)/\rho_j^2 \geq \left(1 - \frac{\beta_+ + \varepsilon - \gamma}{m_+ - \gamma}\right) - C/\rho_j^2.$$

Consequently,

$$\liminf G(w_j)/\rho_j^2 \geq \left(1 - \frac{\beta_+ + \varepsilon - \gamma}{m_+ - \gamma}\right) > 0.$$

This proves (13.31). A similar argument proves (13.32). Thus, $G(v + w)$ is convex in w, is concave in v and satisfies (13.31), (13.32). The theorem now follows from Theorems 13.6, 13.7, Corollary 13.8, and Lemma 13.11. □

Next, we prove Theorem 13.13.

Proof. First, we note that the hypotheses of Theorem 13.13 imply those of Theorem 13.14. In fact, we can take $\beta_\pm = \theta_\pm$, for we have

$$P(x,t,\xi) = \int_0^\xi s[p(x,t,s) - p(x,t,0)]\frac{ds}{s} + p(x,t,0)\xi.$$

Hence,

$$\theta_-\xi^2 \le 2P(x,t,\xi) - 2p(x,t,0)\xi \le \theta_+\xi^2$$

by (13.26). Thus, there exists a weak solution of (13.1)–(13.3). To show that it is unique, we note that for fixed $v \in N$, $G(v+w)$ is strictly convex in w. Hence, if there were two distinct points such that $G'(u_0) = G'(u) = 0$, we would have by Lemma 13.10 both $G(u) < G(u_0)$ and $G(u_0) < G(u)$, an impossibility. □

13.8 Notes and remarks

The problem (13.1)–(13.3) has been considered by Smiley [147] and Mawhin [90]. They assume

$$\beta_0\xi^2 \le \xi[p(x,t,\xi_1) - p(x,t,\xi_0)], \quad \xi = \xi_1 - \xi_0 \in \mathbb{R},$$

and

$$|p(x,t,\xi_1) - p(x,t,\xi_0)| \le \beta_1|\xi|,$$

where $0 < \beta_1$ and $\beta_1^2 < \beta_0$. This implies $0 < \beta_0 \le \beta_1 < 1$ and $\beta_1 - \beta_0 < \frac{1}{2}(1-\beta_0)$. Since $\rho(\Box)$ consists only of integers, our results are extensions of their results. Theorems 13.13 and 13.14 were proved in [125].

Chapter 14

Type (II) Regions

14.1 Introduction

The Fučík spectrum described in Chapter 11 arises in the study of semilinear elliptic boundary-value problems of the form

$$Au = f(x, u), \tag{14.1}$$

where A is a selfadjoint operator having compact resolvent on $L^2(\Omega)$, $\Omega \subset \mathbb{R}^n$, and $f(x, t)$ is a Carathéodory function on $\bar{\Omega} \times \mathbb{R}$ such that

$$\begin{aligned} f(x, t)/t \quad &\to \quad a \text{ a.e. as } t \to -\infty \\ &\to \quad b \text{ a.e. as } t \to +\infty. \end{aligned} \tag{14.2}$$

If $|u(x)|$ is large, then (14.1) approximates the equation

$$Au = bu^+ - au^-, \tag{14.3}$$

where $u^\pm = \max\{\pm u, 0\}$. It was first noticed by Fučík [68] that (14.3) plays an important part in the study of (14.1) when (14.2) holds. Fučík studied the problem

$$-u'' = bu^+ - au^- \text{ in } (0, \pi), \quad u(0) = u(\pi) = 0 \tag{14.4}$$

and showed that there is a substantial difference in the solvability of (14.1) if (14.3) has nontrivial solutions. We now call the set Σ of those $(a, b) \in \mathbb{R}^2$ for which (14.3) has nontrivial solutions the Fučík spectrum of A. Fučík showed that for (14.4), Σ consists of a sequence of decreasing curves passing through the points (k^2, k^2), $k = 1, 2, \ldots,$ with one or two curves emanating from each of these points. Points not on these curves are not in Σ. He also noticed that there are two different types of regions between the curves, namely,

(I) regions between curves passing through consecutive points

$$(k^2, k^2), ([k + 1]^2, [k + 1]^2)$$

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and

(II) regions between curves passing through the same point (k^2, k^2).

In the type (I) regions one can solve

$$-u'' = bu^+ - au^- + p(x), \quad x \in (0,\pi),\ u(0) = u(\pi) = 0, \tag{14.5}$$

for arbitrary $p(x) \in L^2(0,\pi)$, while this is not so for regions of type (II).

No complete description of Σ for the general case (14.1) has been found. If

$$0 < \lambda_0 < \lambda_1 < \cdots < \lambda_k < \cdots \tag{14.6}$$

are the eigenvalues of A, it was shown in [111] and [122] that in the square

$$[\lambda_{\ell-1}, \lambda_{\ell+1}]^2,$$

there are decreasing curves $C_{\ell,1}, C_{\ell,2}$ (which may coincide) passing through the point

$$(\lambda_\ell, \lambda_\ell)$$

such that all points above or below both curves in the square are not in Σ [and are of type (I)] while points on the curves are in Σ. The status of points between the curves (when they do not coincide) is unknown in general. However, it was shown in [72] that when λ_ℓ is a simple eigenvalue, points between the curves are not in Σ. On the other hand, C. and W. Margulies [91] have shown that there are boundary-value problems for which many curves in Σ emanate from a point $(\lambda_\ell, \lambda_\ell)$ when λ_ℓ is a multiple eigenvalue. (Clearly, these curves are contained in the region between the curves $C_{\ell,1}$ and $C_{\ell,2}$.)

As expected, the boundary-value problem (14.1) is more readily solved when (a, b) is in a region of type (I) (i.e., on the same side of both curves $C_{\ell 1}, C_{\ell 2}$). For such points, the following was proved in [111] and [122].

Theorem 14.1. *Assume that $f(x,t)$ is of the form*

$$f(x,t) = bt^+ - at^- + p(x,t), \tag{14.7}$$

where $p(x,t)$ satisfies

$$|p(x,t)| \le V(x)^{1-\sigma}|t|^\sigma + W(x), \tag{14.8}$$

with $0 \le \sigma < 1$ and $V, W \in L^2(\Omega)$. Then (14.1) has a solution. In particular,

$$Au = bu^+ - au^- + p(x) \tag{14.9}$$

has a solution for each $p \in L^2(\Omega)$.

However, when it comes to points on the curves $C_{\ell,1}, C_{\ell,2}$, no such theorem holds. In Chapter 11 we addressed this issue and presented sufficient conditions for the existence of solutions of (14.1) when (a, b) is on either $C_{\ell,1}$ or $C_{\ell,2}$. [Needless to

say, much more is required of $p(x,t)$ in Theorems 11.1 and 11.2 than in Theorem 14.1.] In the present chapter we consider the situation when (a,b) is in the region between the curves $C_{\ell,1}, C_{\ell,2}$ (when they do not coincide). As we saw earlier, such points may or may not belong to Σ. We shall be concerned with those points in this region that are not in Σ. Even in the case when none of these points is in Σ (as in the case of a simple eigenvalue), we cannot prove a theorem as comprehensive as Theorem 14.1. However, we can prove the theorems below. We assume that

$$|p(x,t)| \le C(|t|+1), \quad x \in \Omega, t \in \mathbb{R}, \tag{14.10}$$

and that $E(\lambda_\ell)$, the subspace of eigenfunctions corresponding to λ_ℓ, is contained in $L^\infty(\Omega)$ (i.e., the eigenfunctions corresponding to λ_ℓ are bounded). We define

$$F(x,t) = \int_0^t f(x,s)\,ds \tag{14.11}$$

and

$$P(x,t) = \int_0^t p(x,s)\,ds. \tag{14.12}$$

We have

Theorem 14.2. *Let (a,b) be a point in $Q_\ell := (\lambda_{\ell-1}, \lambda_{\ell+1})^2$ that lies below the upper curve $C_{\ell,2}$. Assume (14.2), (14.7), and*

$$f(x,t_1) - f(x,t_0) > \lambda_{\ell-1}(t_1 - t_0), \quad t_0 < t_1,\ x \in \Omega, \tag{14.13}$$

$$2P(x,t) \le W_1(x) \in L^1(\Omega), \quad x \in \Omega,\ t \in \mathbb{R}, \tag{14.14}$$

$$2F(x,t) \le \lambda_{\ell+1} t^2, \quad x \in \Omega,\ t \in \mathbb{R}, \tag{14.15}$$

$$\lambda_\ell t^2 \le 2F(x,t), \quad |t| < \delta \text{ for some } \delta > 0. \tag{14.16}$$

If (a,b) is not in Σ, then (14.1) has a nontrivial solution.

Theorem 14.3. *Let (a,b) be a point in $Q_\ell \backslash \Sigma$ that lies above the lower curve $C_{\ell,1}$. Assume (14.2), (14.7), and*

$$2P(x,t) \ge -W_1(x) \in L^1(\Omega), \quad x \in \Omega,\ t \in \mathbb{R}, \tag{14.17}$$

$$f(x,t_1) - f(x,t_0) < \lambda_{\ell+1}(t_1 - t_0), \quad t_0 < t_1,\ x \in \Omega, \tag{14.18}$$

$$\lambda_{\ell-1} t^2 \le 2F(x,t), \quad x \in \Omega,\ t \in \mathbb{R}, \tag{14.19}$$

$$2F(x,t) \le \lambda_\ell t^2, \quad |t| < \delta \text{ for some } \delta > 0. \tag{14.20}$$

Then (14.1) has a nontrivial solution.

It should be noted that these theorems hold even when λ_ℓ is a multiple eigenvalue. In proving the theorems, we examine the functional

$$G(u) = \|u\|_D^2 - 2\int_\Omega F(x,u)dx, \quad u \in D = D(A^{1/2}), \tag{14.21}$$

where $\|u\|_D = \|A^{1/2}u\|$, and search for solutions of $G'(u) = 0$. Even though (a, b) is in a region of type (II), we can find a manifold S on which G is bounded from below. Unfortunately, S has a boundary that cannot be linked with a subspace, and one must search for another manifold that links the boundary of S and on which G is bounded from above. Once this is achieved, we can apply the theorems of Chapter 2 to obtain a Palais–Smale sequence. We then show that there is a convergent subsequence due to the fact that (a, b) is not in Σ. In Theorem 14.3 we reverse the procedure, finding a manifold on which G is bounded from above and then searching for a linking set on which it is bounded from below.

14.2 The asymptotic equation

In this section we show how information concerning (14.3) affects the solvability of (14.1). For each fixed positive integer ℓ, we let N_ℓ denote the subspace of $D = D(A^{1/2})$ spanned by the eigenfunctions of A corresponding to the eigenvalues $\lambda_0, \lambda_1, \dots, \lambda_\ell$, and we let $M_\ell = N_\ell^\perp \cap D$. Then $D = M_\ell \oplus N_\ell$. For $(a, b) \in \mathbb{R}^2$, we define

(14.22) $$\|u\|_D^2 = (Au, u),$$

(14.23) $$I(u, a, b) = \|u\|_D^2 - a\|u^-\|^2 - b\|u^+\|^2, \quad u \in D,$$

(14.24) $$M_\ell(a, b) = \inf_{\substack{w \in M_\ell \\ \|w\|_D = 1}} \sup_{v \in N_\ell} I(v + w, a, b),$$

(14.25) $$m_\ell(a, b) = \sup_{\substack{v \in N_\ell \\ \|v\|_D = 1}} \inf_{w \in M_\ell} I(v + w, a, b),$$

(14.26) $$\nu_\ell(a) = \sup\{b : M_\ell(a, b) \ge 0\},$$

(14.27) $$\mu_\ell(a) = \inf\{b : m_\ell(a, b) \le 0\}.$$

It follows from Lemma 11.3 that $\nu_{\ell-1}(a) \le \mu_\ell(a)$ in Q_ℓ. We let $C_{\ell,1}$ be the (lower) curve $b = \nu_{\ell-1}(a)$ and we let $C_{\ell,2}$ be the (upper) curve $b = \mu_\ell(a)$. Thus, by (14.27), if $(a, b) \in Q_\ell$ is below the curve $C_{\ell,2}$, then

(14.28) $$m_\ell(a, b) > 0.$$

Consequently, there is a $y_0 \in N_0 = E(\lambda_\ell)$ such that

(14.29) $$\sup_{v \in N_{\ell-1}} \inf_{w \in M_\ell} I(v + w + y_0, a, b) > 0.$$

Note that $I(v + w + y_0, a, b)$ is strictly convex and lower semi-continuous in $w \in M_\ell$ and strictly concave and continuous in $v \in N_{\ell-1}$ because $(a, b) \in Q_\ell$. Moreover,

$$I(w + y_0, a, b) \to \infty \text{ as } \|w\|_D \to \infty, \quad w \in M.$$

To see this, let $\|w\|_D = \rho \to \infty$. Then, for $u = (w + y_0)/\rho$, we have

$$I(u, a, b) \ge 1 - a\|u^-\|^2 - b\|u^+\|^2 \to 1 - a\|\tilde{w}^-\|^2 - b\|\tilde{w}^+\|^2 > 0,$$

where $\|\tilde{w}\| \leq 1$, since $a, b < \lambda_{\ell+1}$. Similarly,

$$I(v + y_0, a, b) \to \infty \text{ as } \|v\|_D \to \infty, \quad v \in N.$$

Hence we may apply Theorem 13.6 to conclude that there are unique $v_0 \in N_{\ell-1}$, $w_0 \in M_\ell$ such that

$$\begin{aligned} I(v_0 + w_0 + y_0, a, b) &= \sup_{v \in N_{\ell-1}} \inf_{w \in M_\ell} I(v + w + y_0, a, b) \\ &= \inf_{w \in M_\ell} \sup_{v \in N_{\ell-1}} I(v + w + y_0, a, b). \end{aligned} \tag{14.30}$$

In particular, this shows that $y_0 \neq 0$, for otherwise $v_0 = 0, w_0 = 0$ would satisfy (14.30) and the uniqueness would violate (14.29). We also see from (14.29) and (14.30) that

$$\sup_{v \in N_{\ell-1}} I(v + w + y_0, a, b) > 0, \quad w \in M_\ell. \tag{14.31}$$

Moreover, it follows that there is a continuous map θ from $M_{\ell-1}$ to $N_{\ell-1}$ such that

$$\|\theta(w)\| \leq C\|w\|_D, \quad w \in M_{\ell-1}, \tag{14.32}$$

$$\theta(sw) = s\theta(w), \quad s \geq 0, \tag{14.33}$$

and $\tilde{v} = \theta(w)$ is the only solution of

$$I'(w + \tilde{v}, a, b) \perp N_{\ell-1}, \quad w \in M_{\ell-1}, \tag{14.34}$$

and

$$I(w + \tilde{v}, a, b) = \max_{v \in N_{\ell-1}} I(w + v, a, b), \quad w \in M_{\ell-1}. \tag{14.35}$$

Thus, (14.31) implies

$$I(w + sy_0 + \theta(w + sy_0), a, b) \geq 0, \quad w \in M_\ell,\ s \geq 0. \tag{14.36}$$

Another way of writing (14.36) is

$$I(u, a, b) \geq 0, \quad u \in S_{\ell 2} \tag{14.37}$$

where

$$S_{\ell 2}, = \{u \in D : I'(u, a, b) \perp N_{\ell-1}, (u, y_0) \geq 0\}. \tag{14.38}$$

Thus, we have proved

Lemma 14.4. *If $(a, b) \in Q_\ell$ is below the upper curve $C_{\ell,2}$, then there is a $y_0 \in E(\lambda_\ell)\backslash\{0\}$ such that (14.37) holds for all $u \in S_{\ell 2}$, where $S_{\ell 2}$ is given by (14.38).*

In the same vein, if $(a,b) \in Q_\ell$ is above the lower curve $C_{\ell 1}$, then there is a $y_1 \in N_0$ such that

$$\inf_{w \in M_\ell} \sup_{v \in N_{\ell-1}} I(v + w + y_1, a, b) < 0. \tag{14.39}$$

Using the same reasoning as before, we have $y_1 \neq 0$ and (14.30) holds with y_0 replaced by y_1. This implies

$$\inf_{w \in M_\ell} I(v + w + y_1, a, b) < 0, \quad v \in N_{\ell-1}. \tag{14.40}$$

Now we use the fact that there is a continuous map τ from N_ℓ to M_ℓ such that

$$\|\tau(v)\|_D \leq C\|v\|, \quad v \in N_\ell, \tag{14.41}$$

$$\tau(sv) = s\tau(v), \quad s \geq 0, \tag{14.42}$$

and $\tilde{w} = \tau(v)$ is the unique solution of

$$I'(v + \tilde{w}, a, b) \perp M_\ell, \quad v \in N_\ell, \tag{14.43}$$

and

$$I(v + \tilde{w}, a, b) = \inf_{w \in M_\ell} I(v + w, a, b), \quad v \in N_\ell. \tag{14.44}$$

Thus, (14.40) implies

$$I(v + sy_1 + \tau(v + sy_1), a, b) \leq 0, \quad v \in N_{\ell-1}, \ s \geq 0, \tag{14.45}$$

or

$$I(u, a, b) \leq 0, \quad u \in S_{\ell 1}, \tag{14.46}$$

where

$$S_{\ell 1} = \{u \in D : I'(u, a, b) \perp M_\ell, (u, y_1) \geq 0\}. \tag{14.47}$$

We therefore have

Lemma 14.5. *If $(a,b) \in Q_\ell$ lies above the lower curve $C_{\ell,1}$, then there is a $y_1 \in E(\lambda_\ell)\backslash\{0\}$ such that (14.46) holds for all $u \in S_{\ell 1}$, where $S_{\ell 1}$ is given by (14.47).*

14.3 Local estimates

In proving Theorems 14.2 and 14.3, we shall make use of the functional

$$G(u) = \|u\|_D^2 - 2\int_\Omega F(x,u)\,dx = I(u,a,b) - 2\int_\Omega P(x,u)\,dx \tag{14.48}$$

for functions $u \in D$. When (14.10) holds, it is easily checked that $G \in C^1(D, \mathbb{R})$ and that $u \in D$ is a solution of (14.1) if, and only if, it satisfies

$$G'(u) = 0. \tag{14.49}$$

In this section we shall examine some properties of G under the hypotheses of the theorems. First, we have

Lemma 14.6. *If (14.16) holds, then for each $\rho > 0$ sufficiently small, there is a positive ε such that*

$$G(v + y) \leq -\varepsilon\|v\|^2, \quad v \in N_{\ell-1},\ y \in E(\lambda_\ell),\ \|v + y\| \leq \rho. \tag{14.50}$$

Proof. Since $E(\lambda_\ell) \subset L^\infty(\Omega)$, there is a $\rho > 0$ such that

$$\|y\|_D \leq \rho \text{ implies } |y(x)| \leq \delta/2, \quad y \in E(\lambda_\ell), \tag{14.51}$$

where δ is the constant given in (14.16). Let $w = v + y$, where $v \in N_{\ell-1}$, $y \in E(\lambda_\ell)$. If

$$\|w\|_D \leq \rho \quad \text{and} \quad |w(x)| \geq \delta, \tag{14.52}$$

then

$$\delta \leq |w(x)| \leq |v(x)| + |y(x)| \leq |v(x)| + \delta/2. \tag{14.53}$$

Consequently, (14.52) implies

$$|y(x)| \leq \delta/2 \leq |v(x)| \tag{14.54}$$

and

$$|w(x)| \leq 2|v(x)|. \tag{14.55}$$

By (14.10),

$$|F(x,t)| \leq C(t^2 + |t|). \tag{14.56}$$

Thus,

$$\begin{aligned} G(w) &\leq \|w\|_D^2 - \lambda_\ell \int_{|w|<\delta} w^2 dx + C \int_{|w|>\delta} (w^2 + |w|) dx \\ &\leq \|w\|_D^2 - \lambda_\ell \|w\|^2 + C' \int_{|w|>\delta} (w^2 + |w|) dx \\ &\leq \|v\|_D^2 - \lambda_\ell \|v\|^2 + 4C'(1 + \delta^{-1}) \int_{2|v|>\delta} v^2 dx \\ &\leq \|v\|_D^2 - \lambda_\ell \|v\|^2 + C'' \int |v|^\sigma dx \\ &\leq \left(1 - \frac{\lambda_\ell}{\lambda_{\ell-1}} + C''' \|v\|_D^{\sigma-2}\right) \|v\|_D^2, \end{aligned} \tag{14.57}$$

where $\sigma > 2$. If we take ρ sufficiently small, this implies (14.50). □

Lemma 14.7. *If (14.20) holds, then for each $\rho > 0$ sufficiently small, there is an $\varepsilon > 0$ such that*

$$G(w + y) \geq \varepsilon\|w\|^2, \quad w \in M_\ell,\ y \in E(\lambda_\ell),\ \|w + y\| \leq \rho. \tag{14.58}$$

The proof of Lemma 14.7 is similar to that of Lemma 14.6 and is omitted.

Lemma 14.8. *Let θ be a continuous map from $M_{\ell-1}$ to $N_{\ell-1}$ satisfying (14.32), and define*

$$H(v+w) = v + w + \theta(w), \quad v \in N_{\ell-1},\ w \in M_{\ell-1}. \tag{14.59}$$

If (14.16) holds and G is given by (14.48), then we have the following alternative: Either (a) for each $\rho > 0$ sufficiently small, there is an $\varepsilon > 0$ such that

$$G(Hv) \le -\varepsilon, \quad v \in N_\ell \cap \partial B_\rho, \tag{14.60}$$

or (b) there is a $y \in E(\lambda_\ell)\backslash\{0\}$ such that

$$Ay = \lambda_\ell y = f(x, y). \tag{14.61}$$

Proof. For $v \in N_\ell$, write $v = v' + y$, where $v' \in N_{\ell-1}$ and $y \in E(\lambda_\ell)$. Then

$$Hv = v + \theta(y), \quad \|Hv\| \le C\|v\|. \tag{14.62}$$

Thus, by Lemma 14.6, for each $\rho > 0$ sufficiently small, there is an $\varepsilon > 0$ such that

$$G(Hv) \le -\varepsilon\|v' + \theta(y)\|^2, \quad v \in N_\ell \cap B_\rho. \tag{14.63}$$

In particular, $G(Hv) \le 0$ for such v. Assume that option (a) does not hold. Then there is a sequence $v_k = v'_k + y_k$, $v'_k \in N_{\ell-1}$, $y_k \in E(\lambda_\ell)$ such that

$$G(Hv_k) \to 0, \quad \|v_k\| = \rho. \tag{14.64}$$

Since $Hv_k = v_k + \theta(u_k)$, we see from (14.63) that $v'_k + \theta(y_k) \to 0$. Moreover, there is a renamed subsequence such that $y_k \to y_0$ in $E(\lambda_\ell)$. By continuity, $\theta(y_k) \to \theta(y_0)$ and $v'_k \to -\theta(y_0)$. If $y_0 = 0$, then $v_k \to 0$, contradicting the fact that $\|v_k\| = \rho$. Thus, $y_0 \ne 0$ and $Hv_k \to y_0$. This implies

$$G(y_0) = 0. \tag{14.65}$$

If $\rho > 0$ is such that (14.51) holds, then (14.16) implies that

$$\lambda_\ell y_0(x)^2 \le 2F(x, y_0(x)), \quad x \in \Omega. \tag{14.66}$$

But (14.65) says

$$\int_\Omega \{\lambda_\ell y_0(x)^2 - 2F(x, y_0(x))\}dx = 0. \tag{14.67}$$

This together with (14.66) implies that

$$\lambda_\ell y_0(x)^2 \equiv F(x, y_0(x)), \quad x \in \Omega. \tag{14.68}$$

Let $\zeta(x)$ be any function in $C_0^\infty(\Omega)$. Then, for $t > 0$ sufficiently small,

$$t^{-1}\lambda_\ell[(y_0 + t\zeta)^2 - y_0^2] \le t^{-1}2[F(x, y_0 + t\zeta) - F(x, y_0)]. \tag{14.69}$$

Taking the limit as $t \to 0$, we have

$$\lambda_\ell y_0(x)\zeta(x) \le f(x, y_0)\zeta(x), \quad x \in \Omega. \tag{14.70}$$

From this we conclude that

$$\lambda_\ell y_0(x) \equiv f(x, y_0(x)), \quad x \in \Omega. \tag{14.71}$$

Since $y_0 \in E(\lambda_\ell)$, this implies that y_0 satisfies (14.61). $\square$

Lemma 14.9. *Let τ be a continuous map from N_ℓ to M_ℓ such that (14.41) holds and define*

$$H(v + w) = v + w + \tau(v), \quad v \in N_\ell,\ w \in M_\ell. \tag{14.72}$$

If $F(x, t)$ satisfies (14.20), then the following alternative holds: Either (a) for each $\rho > 0$ sufficiently small, there is an $\varepsilon > 0$ such that

$$G(Hw) \ge \varepsilon, \quad w \in M_{\ell-1} \cap \partial B_\rho, \tag{14.73}$$

or (b) there is a $y \in E(\lambda_\ell)\backslash\{0\}$ such that (14.61) holds.

Proof. The proof is similar to that of Lemma 14.8. If $w = w' + y$, $w' \in M_\ell$, $y \in E(\lambda_\ell)$, then $Hw = w + \tau(y)$, $\|Hw\|_D \le C\|w\|_D$ and

$$G(Hw) \ge \varepsilon\|w' + \tau(y)\|^2, \quad w \in M_{\ell-1} \cap B_\rho, \tag{14.74}$$

for ρ sufficiently small by Lemma 14.7. If (a) does not hold and $w_k = w'_k + y_k$, $w'_k \in M_\ell$, $y_k \in E(\lambda_\ell)$, $w_k \in \partial B_\rho$, and $G(Hw_k) \to 0$, then $w'_k + \tau(y_k) \to 0$, $y_k \to y_1$, $\tau(y_k) \to \tau(y_1)$, and $w'_k \to -\tau(y_1)$ for a renamed subsequence. If $y_1 = 0$, then $w_k \to 0$, contradicting the fact that $\|w_k\|_D = \rho$. Now $Hw_k \to y_1$ and

$$G(y_1) = 0. \tag{14.75}$$

If ρ is sufficiently small, (14.51) holds and

$$2F(x, y_1(x)) \le \lambda_\ell y_1(x)^2, \quad x \in \Omega, \tag{14.76}$$

by (14.20). This together with (14.75) implies that

$$2F(x, y_1(x)) \equiv \lambda_\ell y_1(x)^2, \quad x \in \Omega. \tag{14.77}$$

This and (14.20) imply that

$$t^{-1}2[F(x, y_1 + t\zeta) - F(x, y_1)] \le t^{-1}\lambda_\ell[(y_1 + t\zeta)^2 - y_1^2] \tag{14.78}$$

for t small and $\zeta \in C_0^\infty(\Omega)$. This yields

$$f(x, y_1) \equiv \lambda_\ell y_1(x), \quad x \in \Omega \tag{14.79}$$

and y_1 is a solution of (14.61). $\square$

14.4 The solutions

We can now present the proof of Theorems 14.2 and 14.3. We begin with the proof of Theorem 14.2.

Proof. We shall study G given by (14.48) and look for solutions of (14.49). By (14.13) we have for $w \in M_{\ell-1}, v_0, v_1 \in N_{\ell-1}, v = v_1 - v_0$,

$$\begin{aligned}\frac{1}{2}(G'(w+v_1) - G'(w+v_0), v) \\ &= \|v\|_D^2 - (f(w+v_1) - f(w+v_0), v) \\ &\le \int_\Omega \{\lambda_{\ell-1} v^2 - [f(x, w+v_1) - f(x, w+v_0)]v\} dx.\end{aligned} \tag{14.80}$$

The right-hand side will be negative unless $v \equiv 0$ in view of (14.13). Thus, there is a unique solution $\hat{\theta}(w)$ of

$$G(w + \hat{\theta}(w)) = \sup_{v \in N_{\ell-1}} G(w+v) \tag{14.81}$$

(Lemma 13.5). Moreover, $\hat{\theta}$ is continuous and satisfies (14.32). Define

$$H(v+w) = v + w + \hat{\theta}(w), \quad v \in N_\ell, \ w \in M_{\ell-1}, \tag{14.82}$$

which is continuous as well. Let

$$A_R = [M_\ell \cap \bar{B}_R] \cup \{u = w + sy_0 : w \in M_\ell, s \ge 0, \|u\| = R\} \tag{14.83}$$

where y_0 is the element of $E(\lambda_\ell)\backslash\{0\}$ given in Lemma 14.4. By Example 7 of Section 3.5, A_R links $N_\ell \cap \partial B_\rho$ $[hm]$ whenever $0 < \rho < R$. In view of Proposition 3.11, HA_R links $H[N_\ell \cap \partial B_\rho]$ $[hm]$. Let θ be the map described in Section 14.2 satisfying (14.32)–(14.36). Then, by (14.81),

$$\begin{aligned} G(w + sy_0 + \hat{\theta}(w+sy_0)) &\ge G(w + sy_0 + \theta(w+sy_0)) \\ &= I(w + sy_0 + \theta(w+sy_0), a, b) \\ &\quad -2\int_\Omega P(x, w + sy_0 + \theta(w+sy_0))dx \\ &\ge -\int_\Omega W_1(x)\, dx \\ &\equiv -B_1 \end{aligned} \tag{14.84}$$

for $w \in M_\ell, s \ge 0$. Thus,

$$G(u) \ge -B_1, \quad u \in HA_R. \tag{14.85}$$

We can do a bit better on that portion of A_R contained in M_ℓ. In fact, (14.16) implies

$$\begin{aligned} G(w + sy_0 + \hat{\theta}(w)) &\geq G(w) = \|w\|_D^2 - 2\int_\Omega F(x, w)\,dx \\ &\geq \|w\|_D^2 - \lambda_{\ell+1}\|w\|^2 \\ &\geq 0 \end{aligned} \tag{14.86}$$

for $w \in M_\ell$. On the other hand, we see from Lemma 14.8 that

$$G(u) \leq -\varepsilon < 0, \quad u \in H[N_\ell \cap \partial B_\rho], \tag{14.87}$$

for $\rho > 0$ sufficiently small unless there is a $y \in E(\lambda_\ell)\backslash\{0\}$ satisfying (14.61). But in the latter case the theorem is proved. Thus, we may assume that (14.87) holds. We can now apply Theorem 3.15 to conclude that there is a sequence $\{u_k\} \subset D$ such that

$$G(u_k) \to c, \quad -\infty < c \leq -\varepsilon, \quad G'(u_k) \to 0. \tag{14.88}$$

Thus

$$\|u_k\|_D^2 - 2\int_\Omega F(x, u_k)dx \to c \tag{14.89}$$

and

$$(u_k, v)_D - (f(u_k), v) \to 0, \quad v \in D. \tag{14.90}$$

If

$$\rho_k = \|u_k\|_D \to \infty, \tag{14.91}$$

let $\tilde{u}_k = u_k/\rho_k$. Then $\|\tilde{u}_k\|_D = 1$, and there is a renamed subsequence such that $\tilde{u}_k \to \tilde{u}$ weakly in D, strongly in $L^2(\Omega)$, and a.e. in Ω. By (14.56),

$$|F(x, u_k)|/\rho_k^2 \leq C(\tilde{u}_k^2 + |\tilde{u}_k|/\rho_k) \tag{14.92}$$

and the right-hand side converges to $C\tilde{u}^2$ in $L^1(\Omega)$. Moreover,

$$2F(x, u_k)/\rho_k^2 \to b(\tilde{u}^+)^2 + a(\tilde{u}^-)^2 \text{ as } k \to \infty \text{ a.e. in } \Omega. \tag{14.93}$$

Hence,

$$2\int_\Omega F(x, u_k)dx/\rho_k^2 \to b\|\tilde{u}^+\|^2 + a\|\tilde{u}^-\|^2. \tag{14.94}$$

But the left-hand side of (14.94) converges to 1 by (14.89). Thus, $\tilde{u} \not\equiv 0$. Now, by (14.10),

$$|f(x, u_k)|/\rho_k \leq C(|\tilde{u}_k| + \rho_k^{-1}) \tag{14.95}$$

and the right-hand side converges to $C|\tilde{u}|$ in $L^2(\Omega)$. Since

$$f(x,u_k)/\rho_k \to b\tilde{u}^+ - a\tilde{u}^- \text{ a.e. in } \Omega \text{ as } k \to \infty, \tag{14.96}$$

we have

$$(\tilde{u}, v)_D = (b\tilde{u}^+ - a\tilde{u}^-, v), \quad v \in D, \tag{14.97}$$

by (14.90). This says that $\tilde{u}$ satisfies (14.3). Since $(a,b) \notin \Sigma$, we must have $\tilde{u} \equiv 0$, contradicting the conclusion reached earlier. Thus, (14.91) cannot hold. But then standard arguments imply that (14.1) has a solution u satisfying $G(u) = c \le -\varepsilon$. This shows that $u \not\equiv 0$, and the proof is complete. $\square$

Proof of Theorem 14.3. In this case we take

$$A_R = [N_{\ell-1} \cap \bar{B}_R] \cup \{u = v + sy_1 : v \in N_{\ell-1}, s \ge 0, \|u\| = R\}, \tag{14.98}$$

where y_1 is the element of $E(\lambda_\ell)\backslash\{0\}$ given by Lemma 14.5. By (14.18), there is a continuous map $\hat{\tau}$ from N_ℓ to M_ℓ such that (14.41) holds (for $\hat{\tau}$) and

$$G(v + \hat{\tau}(v)) = \inf_{w \in M_\ell} G(v + w). \tag{14.99}$$

If

$$H(v + w) = v + w + \hat{\tau}(v), \quad v \in N_\ell,\ w \in M_\ell, \tag{14.100}$$

then H is a homeomorphism of D onto itself. Thus, HA_R links $H[N_\ell \cap \partial B_\rho]$ for $0 < \rho < R$ (Example 7 of Section 3.5 and Proposition 3.11). Let τ be the map described in Section 14.2 satisfying (14.41)–(14.45). Then we have

$$\begin{aligned} G(v + sy_1 + \hat{\tau}(v + sy_1)) &\le G(v + sy_1 + \tau(v + sy_1)) \\ &= I(v + sy_1 + \tau(v + sy_1), a, b) \\ &\quad -2\int_\Omega P(x, v + sy_1 + \tau(v + sy_1))dx \\ &\le B_1 \end{aligned}$$

for $v \in N_{\ell-1},\ s \ge 0$. This implies

$$G(u) \le B_1, \quad u \in HA_R. \tag{14.101}$$

Moreover, we have

$$\begin{aligned} G(v + \hat{\tau}(v)) &\le G(v) = \|v\|_D^2 - 2\int_\Omega F(x,v)\,dx \\ &\le \|v\|_D^2 - \lambda_{\ell-1}\|v\|^2 \le 0, \quad v \in N_{\ell-1}. \end{aligned}$$

Also, Lemma 14.9 implies that

$$G(w) \ge \varepsilon > 0, \quad w \in H[M_{\ell-1} \cap \partial B_\rho], \tag{14.102}$$

for $\rho > 0$ sufficiently small unless there is a $y \in E(\lambda_\ell)\backslash\{0\}$ satisfying (14.61). The latter option implies the conclusion of the theorem. We may therefore assume that (14.102) holds. We can now apply Theorem 3.15 to conclude that there is a sequence $\{u_k\} \subset D$ such that

$$G(u_k) \to c, \quad \varepsilon \le c < \infty, \quad G'(u_k) \to 0. \tag{14.103}$$

We now follow the proof of Theorem 14.2 to conclude that there is a solution of (14.1) satisfying $G(u) = c \ge \varepsilon$. This implies that $u \not\equiv 0$, and the proof is complete. □

14.5 Notes and remarks

Since the work of Fučík , many authors have studied other problems of the form (14.1) when (14.2) holds (cf., e.g.,[29]–[30], [52], [54], [55], [58], [56], [63], [64], [72], [73], [80], [81], [82], [90], [105], [122]–[116] and the references quoted in them. In [72], [73], [33], [30] the problem (14.1) was considered for $(a, b) \in Q_\ell$ when λ_ℓ is a simple eigenvalue. If φ is a corresponding eigenfunction, then it was shown in [72] that for each $s \in \mathbb{R}$, there are a unique function u_s and a unique constant $C_s(a, b)$ such that

$$Au_s = bu_s^+ - au_s^- + C_s(a, b)\varphi, \quad (u_s, \varphi) = s, \tag{14.104}$$

with

$$\begin{aligned} C_s = C_s(a, b) &= sC_1, \quad && s \ge 0 \\ &= sC_{-1}, \quad && s < 0. \end{aligned}$$

Clearly, $(a, b) \in \Sigma$ iff $C_1C_{-1} = 0$. The region where $C_1 > 0, C_{-1} > 0$ is below both curves, the region $C_1 < 0, C_{-1} < 0$ is above the curves, and the region $C_1C_{-1} < 0$ is between the curves. It is in this sense that problems for regions of type (II) were considered by these authors for simple eigenvalues.

Theorems 14.2 and 14.3 were given in [116].

Chapter 15

Weak Sandwich Pairs

15.1 Introduction

In Chapter 7 we discussed the situation in which one cannot find linking sets that separate the functional, i.e., satisfy (7.1). Are there sets such that the opposite of (7.1) will imply (3.31)? More precisely, are there sets A, B such that (7.3) implies that there is a sequence satisfying (7.4)? This was answered in the affirmative in Chapter 7. Such pairs exist. This has led to

Definition 15.1. *We say that a pair of subsets A, B of a Banach space E forms a sandwich, if, for any $G \in C^1(E, \mathbb{R})$, inequality (7.3) implies the existence of a PS sequence (7.4).*

It follows from Theorem 3.17 that M, N form a sandwich pair if one of them is finite-dimensional. (Note that $m_0 \leq m_1$.) This is a severe drawback in many applications.

The purpose of the present chapter is to find a counterpart of sandwich pairs that deals with the case when both sets in the pair are infinite-dimensional. In order to do this, we required weak-to-weak continuous differentiability of the functional as we did in Theorem 15.2. We call such pairs **weak sandwiches.**

The purpose of this chapter is to solve systems of equations of the form

$$Av = f(x, v, w) \tag{15.1}$$

$$Bw = g(x, v, w), \tag{15.2}$$

where A, B are linear partial differential operators. The variational approach to solving such a system is to study a functional $G(u)$ chosen so that the system is equivalent to

$$G'(u) = 0. \tag{15.3}$$

(In very many cases, such a functional can be found.) The sandwich theorem, Theorem 3.17, is very useful in dealing with equations or systems for which the corresponding functional is semibounded in one of the directions only on a subspace of finite

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dimension. However, there are many systems for which this is not the case. On the other hand, the theorem is probably not true if both subspaces are infinite-dimensional. In the present chapter we shall show that the theorem is indeed true if we require more than mere continuous differentiability of the functional. The requirement we have chosen is present in many applications. It is the weak-to-weak continuous differentiability defined in Chapter 10 (cf. Definition 10.1). For such functionals, we have

Theorem 15.2. *Let N be a closed subspace of a Hilbert space E and let $M = N^\perp$. Let G be a weak-to-weak continuously differentiable functional on E such that*

$$m_0 := \inf_{w \in M} G(w) \neq -\infty \tag{15.4}$$

and

$$m_1 := \sup_{v \in N} G(v) \neq \infty. \tag{15.5}$$

Then there are a constant $c \in \mathbb{R}$ and a sequence $\{u_k\} \subset E$ such that

$$G(u_k) \to c, \quad m_0 \le c \le m_1, \quad quG'(u_k) \to 0. \tag{15.6}$$

We shall prove Theorem 15.2 in the next section, where we introduce weak sandwich pairs. Applications will be given in Section 15.3.

15.2 Weak sandwich pairs

We now introduce the corresponding definition for the case when both sets A, B are infinite-dimensional.

Definition 15.3. *We shall say that a pair of subsets A, B of a Banach space E forms a* **weak sandwich pair** *if, for any weak-to-weak continuously differentiable $G \in C^1(E, \mathbb{R})$, the inequality*

$$-\infty < b_0 := \inf_B G \le a_0 := \sup_A G < \infty \tag{15.7}$$

implies that there is a sequence $\{u_k\}$ satisfying

$$G(u_k) \to c, \quad b_0 \le c \le a_0, \quad G'(u_k) \to 0. \tag{15.8}$$

We have

Theorem 15.4. *Let E be a separable Hilbert space, let N be a closed subspace of E, and let p be any point of N. Let F be a Lipschitz continuous map of E onto N such that $F|_N = I$,*

$$\|F(g) - F(h)\| \le K\|g - h\|, \quad g, h \in E, \tag{15.9}$$

and, for each finite-dimensional subspace S of E containing p such that $FS \neq \{0\}$, there is a finite-dimensional subspace $S_0 \neq \{0\}$ of N containing p such that

$$v \in S_0,\ w \in S \Longrightarrow F(v + w) \in S_0. \tag{15.10}$$

Then $A = N$, $B = F^{-1}(p)$ form a weak sandwich pair.

Proof. Assume that the theorem is false. Let G be a weak-to-weak continuously differentiable functional on E satisfying (15.7), where A, B are the subsets of E specified in the theorem, such that there is no sequence satisfying (15.8). Then there is a positive number $\delta > 0$ such that

$$\|G'(u)\| \geq 2\delta \tag{15.11}$$

whenever u belongs to the set

$$\hat{E} = \{u \in E : b_0 - 2\delta \leq G(u) \leq a_0 + 2\delta\}. \tag{15.12}$$

Since E is separable, we can norm it with a norm $|u|_w$ satisfying

$$|u|_w \leq \|u\|, \quad u \in E, \tag{15.13}$$

and such that the topology induced by this norm is equivalent to the weak topology of E on bounded subsets of E.

This can be done as follows. Let $\{e_k\}$ be an orthonormal basis for E. We then set

$$|u|_w^2 = \sum_{k=1}^{\infty} \frac{|(u, e_k)|^2}{k^2}.$$

We denote E equipped with this norm by $\tilde{E}$. For $u \in \hat{E}$, let $h(u) = G'(u)/\|G'(u)\|$. Then, by (15.11),

$$(G'(u), h(u)) \geq 2\delta, \quad u \in \hat{E}. \tag{15.14}$$

Let

$$\begin{aligned} T &= (a_0 - b_0 + 4\delta)/\delta, \\ B_R &= \{u \in E : \|u\| < R\}, \\ R &= \sup_{\Omega} \|u\| + T, \\ \hat{B} &= \bar{B}_R \cap \hat{E}, \end{aligned} \tag{15.15}$$

where Ω is a bounded, open subset of N containing the point p such that

$$\rho(\partial\Omega, p) > KT + \delta, \tag{15.16}$$

and ρ is the distance in E. For each $u \in \hat{B}$, there is an $\tilde{E}$ neighborhood $W(u)$ of u such that

$$(G'(v), h(u)) > \delta, \quad v \in W(u) \cap \hat{B}. \tag{15.17}$$

Otherwise there would be a sequence $\{v_k\} \subset \hat{B}$ such that

$$|v_k - u|_w \to 0 \quad \text{and} \quad (G'(v_k), h(u)) \leq \delta. \tag{15.18}$$

Since $\hat{B}$ is bounded in E, $v_k \to u$ weakly in E and (10.5) implies that

$$(G'(v_k), h(u)) \to (G'(u), h(u)) \geq 2\delta \tag{15.19}$$

in view of (15.14). This contradicts (15.17). Let $\tilde{B}$ be the set $\hat{B}$ with the inherited topology of $\tilde{E}$. It is a metric space, and $W(u) \cap \tilde{B}$ is an open set in this space. Thus, $\{W(u) \cap \tilde{B}\}$, $u \in \tilde{B}$, is an open covering of the paracompact space $\tilde{B}$ (cf., e.g., [77]). Consequently, there is a locally finite refinement $\{W_\tau\}$ of this cover. For each τ, there is an element u_τ such that $W_\tau \subset W(u_\tau)$. Let $\{\psi_\tau\}$ be a partition of unity subordinate to this covering. Each ψ_τ is locally Lipschitz continuous with respect to the norm $|u|_w$ and consequently with respect to the norm of E. Let

$$Y(u) = \sum \psi_\tau(u) h(u_\tau), \quad u \in \tilde{B}. \tag{15.20}$$

Then $Y(u)$ is locally Lipschitz continuous with respect to both norms. Moreover,

$$\|Y(u)\| \leq \sum \psi_\tau(u) \|h(u_\tau)\| \leq 1 \tag{15.21}$$

and

$$(G'(u), Y(u)) = \sum \psi_\tau(u)(G'(u), h(u_\tau)) \geq \delta, \quad u \in \hat{B}. \tag{15.22}$$

For $u \in \bar{\Omega} \cap \hat{E}$, let $\sigma(t)u$ be the solution of

$$\sigma'(t) = -Y(\sigma(t)), \quad t \geq 0, \quad \sigma(0) = u. \tag{15.23}$$

Note that $\sigma(t)u$ will exist as long as $\sigma(t)u$ is in $\hat{B}$. Moreover, it is continuous in (u, t) with respect to both topologies.

Next, we note that if $u \in \bar{\Omega} \cap \hat{E}$, we cannot have $\sigma(t)u \in \hat{B}$ and $G(\sigma(t)u) > b_0 - \delta$ for $0 \leq t \leq T$: for by (15.23), (15.22),

$$dG(\sigma(t)u)/dt = (G'(\sigma), \sigma') = -(G'(\sigma), Y(\sigma)) \leq -\delta \tag{15.24}$$

as long as $\sigma(t)u \in \hat{B}$. Hence, if $\sigma(t)u \in \hat{B}$ for $0 \leq t \leq T$, we would have

$$G(\sigma(T)u) - G(u) \leq -\delta T = -(a_0 - b_0 + 4\delta). \tag{15.25}$$

Thus, we would have $G(\sigma(T)u) < b_0 - 4\delta$. On the other hand, if $\sigma(s)u$ exists for $0 \leq s < T$, then $\sigma(t)u \in \hat{B}$. To see this, note that

$$u - \sigma(t)u = z_t(u) := \int_0^t Y(\sigma(s)u)ds. \tag{15.26}$$

By (15.21),

$$\|z_t(u)\| \leq t. \tag{15.27}$$

Consequently,

$$\|\sigma(t)u\| \leq \|u\| + t < R. \tag{15.28}$$

Thus, $\sigma(t)u \in \hat{B}$. We can now conclude that for each $u \in \bar{\Omega} \cap \hat{E}$, there is a $t \geq 0$ such that $\sigma(s)u$ exists for $0 \leq s \leq t$ and $G(\sigma(t)u) \leq b_0 - \delta$. Let

$$T_u := \inf\{t \geq 0 : G(\sigma(t)u) \leq b_0 - \delta\}, \quad u \in \bar{\Omega} \cap \hat{E}. \tag{15.29}$$

Then $\sigma(t)u$ exists for $0 \leq t \leq T_u$ and $T_u < T$. Moreover, T_u is continuous in u. Define

$$\hat{\sigma}(t)u = \begin{cases} \sigma(t)u, & 0 \leq t \leq T_u, \\ \sigma(T_u)u, & T_u \leq t \leq T, \end{cases}$$

for $u \in \bar{\Omega} \cap \hat{E}$. For $u \in \bar{\Omega} \setminus \hat{E}$, define $\hat{\sigma}(t)u = u, 0 \leq t \leq T$. Then $\hat{\sigma}(t)u$ is continuous in (u, t), and

$$G(\hat{\sigma}(T)u) \leq b_0 - \delta, \quad u \in \bar{\Omega}. \tag{15.30}$$

Let

$$\varphi(v, t) = F\hat{\sigma}(t)v, \quad v \in \bar{\Omega}, \quad 0 \leq t \leq T. \tag{15.31}$$

Then φ is a continuous map of $\bar{\Omega} \times [0, T]$ to N. Let

$$K = \{(u, t) : u = \hat{\sigma}(t)v, \ v \in \bar{Q}, \ t \in [0, T]\}.$$

Then K is a compact subset of $\tilde{E} \times \mathbb{R}$. To see this, let (u_k, t_k) be any sequence in K. Then $u_k = \sigma(t_k)v_k$, where $v_k \in \bar{Q}$. Since Q is bounded, there is a subsequence such that $v_k \to v_0$ weakly in E and $t_k \to t_0$ in $[0, T]$. Since $\bar{Q}$ is convex and bounded, v_0 is in $\bar{Q}$ and $|v_k - v_0|_w \to 0$. Since $\hat{\sigma}(t)$ is continuous in $\tilde{E} \times \mathbb{R}$, we have

$$u_k = \hat{\sigma}(t_k)v_k \rightharpoonup \hat{\sigma}(t_0)v_0 = u_0 \in K.$$

Each $u_0 \in \hat{B}$ has a neighborhood $W(u_0)$ in $\tilde{E}$ and a finite-dimensional subspace $S(u_0)$ such that $Y(u) \subset S(u_0)$ for $u \in W(u_0) \cap \hat{B}$. Since $\hat{\sigma}(t)u$ is continuous in (u, t), for each $(u_0, t_0) \in K$, there are a neighborhood $W(u_0, t_0) \subset \tilde{E} \times \mathbb{R}$ and a finite-dimensional subspace $S(u_0, t_0) \subset E$ such that $\hat{z}_t(u) \subset S(u_0, t_0)$ for $(u, t) \in W(u_0, t_0)$, where

$$\hat{z}_t(u) := u - \hat{\sigma}(t)u = \begin{cases} \int_0^t Y(\hat{\sigma}(s)u)ds, & u \in \hat{E}, \\ 0, & u \notin \hat{E}. \end{cases} \tag{15.32}$$

Since K is compact, there are a finite number of points $(u_j, t_j) \subset K$ such that $K \subset W = \cup W(u_j, t_j)$. Let S be a finite-dimensional subspace of E containing p and all the $S(u_j, t_j)$ and such that $FS \neq \{0\}$. Then, for each $v \in \bar{\Omega}$, we have $\hat{z}_t(v) \in S$. By hypothesis, there is a finite-dimensional subspace $S_0 \neq \{0\}$ of N containing p such that $F(v - \hat{z}_t(v)) \in S_0$ for all $v \in \bar{\Omega} \cap S_0$. We note that $\varphi(u, t)$ maps $\bar{\Omega} \cap S_0 \times [0, T]$ into S_0. For t in $[0, T]$, let $\varphi_t(v) = \varphi(v, t)$. Then

$$\varphi_t(v) \neq p, \quad v \in \partial(\Omega \cap S_0) = \partial\Omega \cap S_0, \quad 0 \leq t \leq T. \tag{15.33}$$

To see this, note that if $v \in \partial\Omega$, then

$$\|v - p\| \leq \|v - F\hat{\sigma}(t)v\| + \|F\hat{\sigma}(t)v - p\|.$$

Hence,

$$\|F\hat{\sigma}(t)v - p\| > KT + \delta - tK > 0, \quad v \in \partial\Omega,\ 0 \le t \le T \tag{15.34}$$

since

$$\|F\hat{\sigma}(t)v - v\| \le K \int_0^t \|\hat{\sigma}'(s)v\|\, ds \le Kt.$$

Thus, (15.33) holds. Consequently, the Brouwer degree $d(\varphi_t, \Omega \cap S_0, p)$ is defined. Since φ_t is continuous, we have

$$d(\varphi_T, \Omega \cap S_0, p) = d(\varphi_0, \Omega \cap S_0, p) = d(I, \Omega \cap S_0, p) = 1. \tag{15.35}$$

Hence, there is a $v \in \Omega$ such that $F\hat{\sigma}(T)v = p$. Consequently, $\hat{\sigma}(T)v \in F^{-1}(p) = B$. In view of (15.7), this implies

$$G(\hat{\sigma}(T)v) \ge b_0,$$

contradicting (15.30). Thus, (15.8) holds, and the proof is complete. □

We can now give the proof of Theorem 15.2.

Proof. We take $A = N, \quad B = M, \quad p = 0$, and $F = P_N$, the projection onto N. If S is a finite-dimensional subspace such that $FS \neq \{0\}$, we take $S_0 = FS$. All of the hypotheses of Theorem 15.4 are satisfied. □

Definition 15.5. *Let E, F be Banach spaces. We shall call a map $J \in C(E, F)$* **weak-to-weak continuous** *if, for each sequence*

$$u_k \to u \text{ weakly in } E, \tag{15.36}$$

there exists a renamed subsequence such that

$$J(u_k) \to J(u) \text{ weakly in } F. \tag{15.37}$$

We have

Proposition 15.6. *If A, B is a weak sandwich pair and J is a weak-to-weak continuous diffeomorphism on the entire space having a derivative $J'(u)$ depending compactly on u and satisfying*

$$\|J'(u)^{-1}\| \le C, \quad u \in E, \tag{15.38}$$

then JA, JB is a weak sandwich pair.

Proof. Let G be a weak-to-weak continuously differentiable functional on E satisfying

$$-\infty < b_0 := \inf_{JB} G \le a_0 := \sup_{JA} G < \infty. \tag{15.39}$$

Let

$$G_1(u) = G(Ju), \quad u \in E.$$

Then

$$(G_1(u), h) = (G'(Ju), J'(u)h).$$

If $u_k \to u$ weakly, then there is a renamed subsequence such that

$$J(u_k) \to J(u) \text{ weakly}; \quad J'(u_k) \to J'(u).$$

Hence,

$$(G_1(u_k), h) \to (G'(Ju), J'(u)h),$$

and G_1 is weak-to-weak continuously differentiable. Moreover,

$$-\infty < b_0 := \inf_{JB} G = \inf_{Ju \in JB} G(Ju) = \inf_B G_1 \tag{15.40}$$
$$\leq a_0 := \sup_{JA} G = \sup_{Ju \in JA} G(Ju) = \sup_A G_1 < \infty.$$

Since $A,\ B$ form a weak sandwich pair, there is a sequence $\{h_k\} \subset E$ such that

$$G_1(h_k) \to c, \quad b_0 \leq c \leq a_0, \quad G_1'(h_k) \to 0. \tag{15.41}$$

If we set $u_k = Jh_k$, this becomes

$$G(u_k) \to c, \quad b_0 \leq c \leq a_0, \quad G'(u_k)J'(h_k) \to 0. \tag{15.42}$$

In view of (15.38), this implies $G'(u_k) \to 0$. Thus, $JA,\ JB$ is a sandwich pair. $\square$

Proposition 15.7. *Let N be a closed subspace of a Hilbert space E with complement $M' = M \oplus \{v_0\}$, where v_0 is an element in E having unit norm, and let δ be any positive number. Let $\varphi(t) \in C^1(\mathbb{R})$ be such that*

$$0 \leq \varphi(t) \leq 1, \quad \varphi(0) = 1,$$

and

$$\varphi(t) = 0, \quad |t| \geq 1.$$

Let

$$F(v + w + sv_0) = v + [s + \delta - \delta\varphi(\|w\|^2/\delta^2)]v_0, \quad v \in N,\ w \in M,\ s \in \mathbb{R}. \tag{15.43}$$

Then $A = N' = N \oplus \{v_0\}$, $B = F^{-1}(\delta v_0)$ forms a weak sandwich pair.

Proof. Define

$$J(v + w + sv_0) = v + w + [s - \delta + \delta\varphi(\|w\|^2/\delta^2)]v_0, \quad v \in N,\ w \in M,\ s \in \mathbb{R}.$$

Then J is a diffeomorphism on E satisfying the hypotheses of Proposition 15.6. Moreover, $A = JN'$ and $B = J[M + \delta v_0]$. Since N' and $M + \delta v_0$ form a weak sandwich pair by Theorem 15.2, we see that A, B also form a weak sandwich pair (Proposition 15.6). $\square$

15.3 Applications

Let A, B be positive, self-adjoint operators on $L^2(\Omega)$ with compact resolvents, where $\Omega \subset \mathbb{R}^n$. Let $F(x, v, w)$ be a Carathéodory function on $\Omega \times \mathbb{R}^2$ such that

$$f(x, v, w) = \partial F/\partial v, \quad g(x, v, w) = \partial F/\partial w \tag{15.44}$$

are also Carathéodory functions satisfying

$$|f(x, v, w)| + |g(x, v, w)| \le C_0(|v| + |w| + 1), \quad v, w \in \mathbb{R}, \tag{15.45}$$

and

$$f(x, ty, tz)/t \to \alpha_+(x)v^+ - \alpha_-(x)v^- + \beta_+(x)w^+ - \beta_-(x)w^-, \tag{15.46}$$

$$g(x, ty, tz)/t \to \gamma_+(x)v^+ - \gamma_-(x)v^- + \delta_+(x)w^+ - \delta_-(x)w^- \tag{15.47}$$

as $t \to +\infty, y \to v, z \to w$, where $a^\pm = \max(\pm a, 0)$. We wish to solve the system

$$Av = -f(x, v, w) \tag{15.48}$$

$$Bw = g(x, v, w). \tag{15.49}$$

Let $\lambda_0(\mu_0)$ be the lowest eigenvalue of $A(B)$. We assume that the only solution of

$$-Av = \alpha_+ v^+ - \alpha_- v^- + \beta_+ w^+ - \beta_- w^-, \tag{15.50}$$

$$Bw = \gamma_+ v^+ - \gamma_- v^- + \delta_+ w^+ - \delta_- w^- \tag{15.51}$$

is $v = w = 0$. Our first result is

Theorem 15.8. *Assume*

$$2F(x, s, 0) \ge -\lambda_0^2 - W_1(x), \quad x \in \Omega,\ t \in \mathbb{R}, \tag{15.52}$$

and

$$2F(x, 0, t) \le \mu_0 t^2 + W_2(x), \quad x \in \Omega,\ t \in \mathbb{R}, \tag{15.53}$$

where $W_i(x) \in L^1(\Omega)$. Then the system (15.48), (15.49) has a solution.

Proof. Let $D = D(A^{1/2}) \times D(B^{1/2})$. Then D becomes a Hilbert space with norm given by

$$\|u\|_D^2 = (Av, v) + (Bw, w), \quad u = (v, w) \in D. \tag{15.54}$$

We define

$$G(u) = b(w) - a(v) - 2\int_\Omega F(x, v, w)dx, \quad u \in D, \tag{15.55}$$

where

$$a(v) = (Av, v), \quad b(w) = (Bw, w). \tag{15.56}$$

Then $G \in C^1(D, \mathbb{R})$ and

$$(G'(u), h)/2 = b(w, h_2) - a(v, h_1) - (f(u), h_1) - (g(u), h_2), \tag{15.57}$$

where we write $f(u)$, $g(u)$ in place of $f(x, v, w)$, $g(x, v, w)$, respectively. It is readily seen that the system (15.48), (15.49) is equivalent to

$$G'(u) = 0. \tag{15.58}$$

We let N be the set of those $(v, 0) \in D$ and M the set of those $(0, w) \in D$. Then M, N are orthogonal closed subspaces such that

$$D = M \oplus N. \tag{15.59}$$

If we define

$$Lu = 2(-v, w), \quad u = (v, w) \in D, \tag{15.60}$$

then L is a self-adjoint, bounded operator on D. Also,

$$G'(u) = Lu + c_0(u), \tag{15.61}$$

where

$$c_0(u) = -(A^{-1} f(u), B^{-1} g(u)) \tag{15.62}$$

is compact on D. This follows from (15.45) and the fact that A and B have compact resolvents. It also follows that G' has weak-to-weak continuity, for if $u_k \to u$ weakly, then $Lu_k \to Lu$ weakly and $c_0(u_k)$ has a convergent subsequence. Now, by (15.53),

$$G(0, w) \geq b(w) - \mu_0 \|w\|^2 - \int_\Omega W_2(x)\, dx, \quad (0, w) \in M. \tag{15.63}$$

Thus,

$$\inf_M G \geq - \int_\Omega W(x)\, dx \equiv b_0. \tag{15.64}$$

On the other hand, (15.52) implies

$$G(v, 0) \leq -a(v) + \lambda_0 \|v\|^2 + \int_\Omega W_1(x)\, dx, \quad (v, 0) \in N. \tag{15.65}$$

Thus,

$$\sup_N G \leq \int_\Omega W_1(x)\, dx \equiv a_0. \tag{15.66}$$

We can now apply Theorem 15.2 to conclude that there is a sequence $\{u_k\} \subset D$ such that (15.8) holds. Let $u_k = (v_k, w_k)$. I claim that

$$\rho_k^2 = a(v_k) + b(w_k) \leq C, \tag{15.67}$$

for assume that $\rho_k \to \infty$, and let $\tilde{u}_k = u_k/\rho_k$. Then there is a renamed subsequence such that $\tilde{u}_k \to \tilde{u}$ weakly in D, strongly in $L^2(\Omega)$, and a.e. in Ω. If $h = (h_1, h_2) \in D$, then

(15.68)
$$(G'(u_k), h)/\rho_k = 2b(\tilde{w}_k, h_2) - 2a(\tilde{v}_k, h_1) - 2(f(u_k), h_1)/\rho_k - 2(g(u_k), h_2)/\rho_k.$$

Taking the limit and applying (15.45)–(15.47), we see that $\tilde{u} = (\tilde{v}, \tilde{w})$ is a solution of (15.50), (15.51). Hence, $\tilde{u} = 0$ by hypothesis. On the other hand, since $a(\tilde{v}_k) + b(\tilde{w}_k) = 1$, there is a renamed subsequence such that $a(\tilde{v}_k) \to \tilde{a}, b(\tilde{w}_k) \to \tilde{b}$ with $\tilde{a} + \tilde{b} = 1$. Thus, by (15.46), (15.47), and (15.57),

$$\begin{aligned}(G'(u_k), (\tilde{v}_k, 0))/2\rho_k &= -a(\tilde{v}_k) - (f(u_k), \tilde{v}_k)/\rho_k \\ &\to -\tilde{a} - \int_\Omega (\alpha_+ \tilde{v}^+ - \alpha_- \tilde{v}^- + \beta_+ \tilde{w}^+ - \beta_- \tilde{w}^-)\tilde{v}\, dx\end{aligned}$$

and

$$\begin{aligned}(G'(u_k), (0, \tilde{w}_k))/2\rho_k &= b(\tilde{w}_k) - (g(u_k), \tilde{w}_k)/\rho_k \\ &\to \tilde{b} - \int_\Omega (\gamma_+ \tilde{v}^+ - \gamma_- \tilde{v}^- + \delta_+ \tilde{w}^+ - \delta_- \tilde{w}^-)\tilde{w}\, dx.\end{aligned}$$

Thus, by (15.8),

$$\tilde{a} = -\int_\Omega (\alpha_+ \tilde{v}^+ - \alpha_- \tilde{v}^- + \beta_+ \tilde{w}^+ - \beta_- \tilde{w}^-)\tilde{v} dx \tag{15.69}$$

and

$$\tilde{b} = \int_\Omega (\gamma_+ \tilde{v}^+ - \gamma_- \tilde{v}^- + \delta_+ \tilde{w}^+ - \delta_- \tilde{w}^-)\tilde{w}\, dx. \tag{15.70}$$

Since one of the two numbers $\tilde{a}, \tilde{b}$ is not zero, we see that we cannot have $\tilde{u} \equiv 0$. This contradiction proves (15.67). Once this is known, we can use the usual procedures to show that there is a renamed subsequence such that $u_k \to u$ in D, and u satisfies (15.58). □

Theorem 15.9. *In addition, assume that the eigenfunctions of λ_0 and μ_0 are bounded and nonzero a.e. in Ω and that there is a $q > 2$ such that*

$$\|w\|_q^2 \le Cb(w), \quad w \in M. \tag{15.71}$$

Assume

$$2F(x, 0, t) \le \mu(x)t^2, \quad x \in \Omega,\ t \in \mathbb{R}, \tag{15.72}$$

where

$$\mu(x) \le\not\equiv \mu_0, \quad x \in \Omega, \tag{15.73}$$

and for some $\delta > 0$,

$$2F(x, s, t) \le \mu_0 t^2 - \lambda_0 s^2, \quad |t| + |s| \le \delta. \tag{15.74}$$

Then the system (15.48), (15.49) has a nontrivial solution.

Proof. Let N' be the orthogonal complement of $N_0 = \{\varphi_0\}$ in N, where φ_0 is the eigenfunction of A corresponding to λ_0. Then $N = N' \oplus N_0$. Let M_0 be the subspace of M spanned by the eigenfunctions of B corresponding to μ_0, and let M' be its orthogonal complement in M. Since N_0 and M_0 are contained in $L^\infty(\Omega)$, there is a positive constant ρ such that

$$a(y) \le \rho^2 \Rightarrow \|y\|_\infty \le \delta/4, \quad y \in N_0, \tag{15.75}$$

$$b(h) \le \rho^2 \Rightarrow \|h\|_\infty \le \delta/4, \quad h \in M_0, \tag{15.76}$$

where δ is the number given in (15.74). If

$$a(y) \le \rho^2, \quad b(w) \le \rho^2, \ |y(x)| + |w(x)| \ge \delta, \tag{15.77}$$

we write $w = h + w'$, $h \in M_0$, $w' \in M'$ and

$$\delta \le |y(x)| + |w(x)| \le |y(x)| + |h(x)| + |w'(x)| \le (\delta/2) + |w'(x)|. \tag{15.78}$$

Thus,

$$|y(x)| + |h(x)| \le \delta/2 \le |w'(x)| \tag{15.79}$$

and

$$|y(x)| + |w(x)| \le 2|w'(x)|. \tag{15.80}$$

Now, by (15.74) and (15.80),

$$\begin{aligned}
G(y, w) &= b(w) - a(y) - 2\int_\Omega F(x, y, w)\,dx \\
&\ge b(w) - a(y) - \int_{|y|+|w|<\delta} \{\mu_0 w^2 - \lambda_0 y^2\} dx \\
&\quad - c_0 \int_{|y|+|w|>\delta} (|y| + |w| + 1)^2\,dx \\
&\ge b(w) - a(y) - \mu_0\|w\|^2 + \lambda_0\|y\|^2 - c_1 \int_{2|w'|>\delta} |w'|^q dx \\
&\ge b(w') - \mu_0\|w'\|^2 - c_2 b(w')^{q/2} \\
&\ge \left(1 - \frac{\mu_0}{\mu_1} - c_2 b(w')^{(q/2)-1}\right) b(w'), \quad a(y) \le \rho^2, \ b(w) \le \rho^2,
\end{aligned}$$

where μ_1 is the next eigenvalue of B after μ_0. If we reduce ρ accordingly, we can find a positive constant ν such that

$$G(y, w) \ge \nu b(w'), \quad a(y) \le \rho^2, b(w) \le \rho^2. \tag{15.81}$$

I claim that either (15.48), (15.49) has a nontrivial solution or there is an $\epsilon > 0$ such that

$$G(y, w) \geq \epsilon, \quad a(y) + b(w) = \rho^2. \tag{15.82}$$

To see this, suppose (15.82) did not hold. Then there would be a sequence $\{y_k, w_k\}$ such that $a(y_k) + b(w_k) = \rho^2$ and $G(y_k, w_k) \to 0$. If we write $w_k = w_k' + h_k, w_k' \in M', h_k \in M_0$, then (15.81) tells us that $b(w_k') \to 0$. Thus, $a(y_k) + b(h_k) \to \rho^2$. Since N_0, M_0 are finite-dimensional, there is a renamed subsequence such that $y_k \to y$ in N_0 and $h_k \to h$ in M_0. By (15.75) and (15.76), $\|y\|_\infty \leq \delta/4$ and $\|h\|_\infty \leq \delta/4$. Consequently, (15.74) implies

$$2F(x, y, h) \leq \mu_0 h^2 - \lambda_0 y^2. \tag{15.83}$$

Since

$$G(y, h) = b(h) - a(y) - 2\int_\Omega F(x, y, h)dx = 0, \tag{15.84}$$

we have

$$\int_\Omega \{2F(x, y, h) + \lambda_0 y^2 - \mu_0 h^2\}dx = 0. \tag{15.85}$$

In view of (15.83), this implies

$$2F(x, y, h) \equiv \mu_0 h^2 - \lambda_0 y^2. \tag{15.86}$$

For $\zeta \in C_0^\infty(\Omega)$ and $t > 0$ small, we have

$$2[F(x, y + t\zeta, h) - F(x, y, h)]/t \leq -\lambda_0[(y + t\zeta)^2 - y^2]/t. \tag{15.87}$$

Taking $t \to 0$, we see that

$$f(x, y, h)\zeta \leq -\lambda_0 y\zeta. \tag{15.88}$$

Since this is true for all $\zeta \in C_0^\infty(\Omega)$, we have

$$f(x, y, h) = -\lambda_0 y = -Ay. \tag{15.89}$$

Similarly,

$$2[F(x, y, h + t\zeta) - F(x, y, h)]/t \leq \mu_0[(h + t\zeta)^2 - h^2]/t, \tag{15.90}$$

and, consequently,

$$g(x, y, h)\zeta \leq \mu_0 h\zeta \tag{15.91}$$

and

$$g(x, y, h) = \mu_0 h = Bh. \tag{15.92}$$

We see from (15.89) and (15.92) that (15.48), (15.49) has a nontrivial solution. Thus, we may assume that (15.82) holds.

Next, we note that there is an $\varepsilon > 0$ depending on ρ such that

$$G(0, w) \geq \varepsilon, \quad b(w) \geq \rho > 0.$$

To see this, suppose that $\{w_k\} \subset M$ is a sequence such that

$$G(0, w_k) \to 0, \quad b(w_k) \geq \rho.$$

If

$$b_k = b(w_k) \leq C,$$

this implies

$$b(w_k) - \mu_0 \|w_k\|^2 \to 0$$

and

$$\int [\mu_0 - \mu(x)] w_k^2 \, dx \to 0$$

since

$$G(0, w) \geq b(w) - \mu_0 \|w\|^2 + \int [\mu_0 - \mu(x)] w^2 dx, \quad w \in M.$$

If we write $w_k = w_k' + h_k, w_k' \in M', h_k \in M_0$ as before, then this tells us that $b(w_k') \to 0$. Since M_0 is finite-dimensional, there is a renamed subsequence such that $h_k \to h$. But the two conclusions above tell us that $h = 0$. Since $b(h) \geq \rho$, we see that $\varepsilon > 0$ exists for any constant C. If the sequence $\{b_k\}$ is not bounded, we take $\tilde{w}_k = w_k / b_k^{1/2}$. Then

$$G(0, w_k)/b_k \geq b(\tilde{w}_k) - \mu_0 \|\tilde{w}_k\|^2 + \int [\mu_0 - \mu(x)] (\tilde{w}_k)^2 dx.$$

Next, we note that there is a $\nu > 0$ such that

$$G(0, w) \geq \nu b(w), \quad w \in M. \tag{15.93}$$

Assuming this for the moment, we see that

$$\inf_B G \geq \varepsilon_1 > 0, \tag{15.94}$$

where

$$B = \{w \in M : b(w) \geq \rho^2\} \cup \{u = (s\varphi_0, w) : s \geq 0, w \in M, \|u\|_D = \rho\}, \tag{15.95}$$

and $\varepsilon_1 = \min\{\varepsilon, \nu\rho^2\}$. By (15.66), there is an $R > \rho$ such that

$$\sup_A G = a_0 < \infty, \tag{15.96}$$

where $A = N$. By Proposition 15.7, A, B form a weak sandwich pair. Moreover, G satisfies (15.7) with $\varepsilon_1 \leq b_0$. Hence, there is a sequence $\{u_k\} \subset D$ such that (15.8)

holds with $c \geq \varepsilon_1$. Arguing as in the proof of Theorem 15.8, we see that there is a $u \in D$ such that $G(u) = c \geq \varepsilon_1 > 0,\ G'(u) = 0$. Since $c \neq 0$ and $G(0) = 0$, we see that $u \neq 0$, and we have a nontrivial solution of the system (15.48), (15.49).

It therefore remains only to prove (15.93). Clearly, $\nu \geq 0$. If $\nu = 0$, then there is a sequence $\{w_k\} \subset M$ such that

$$G(0, w_k) \to 0, \quad b(w_k) = 1. \tag{15.97}$$

Thus, there is a renamed subsequence such that $w_k \to w$ weakly in M, strongly in $L^2(\Omega)$, and a.e. in Ω. Consequently,

$$\int_\Omega [\mu_0 - \mu(x)] w_k^2\, dx \leq 1 - \int_\Omega \mu(x) w_k^2\, dx \leq G(0, w_k) \to 0 \tag{15.98}$$

and

$$1 = \int_\Omega \mu(x) w^2 dx \leq \mu_0 \|w\|^2 \leq b(w) \leq 1, \tag{15.99}$$

which means that we have equality throughout. It follows that we must have $w \in E(\mu_0)$, the eigenspace of μ_0. Since $w \not\equiv 0$, we have $w \neq 0$ a.e. But

$$\int_\Omega [\mu_0 - \mu(x)] w^2 dx = 0 \tag{15.100}$$

implies that the integrand vanishes identically on Ω, and consequently $\mu(x) \equiv \mu_0$, violating (15.73). This establishes (15.93) and completes the proof of the theorem. $\square$

15.4 Notes and remarks

Weak sandwich pairs were introduced in [133].

Chapter 16

Multiple Solutions

16.1 Introduction

Nonlinear problems are characterized by the fact that very often solutions are not unique. However, it is sometimes quite difficult to show that more than one solution exists. In this respect, variational methods can be very helpful. We have chosen some examples that illustrate this point.

16.2 Two examples

Let Ω be a smooth, bounded domain in $\mathbb{R}^n$, and let A be a self-adjoint operator on $L^2(\Omega)$. We assume that

$$C_0^\infty(\Omega) \subset D := D(|A|^{1/2}) \subset H^{T,2}(\Omega) \tag{16.1}$$

holds for some $T > 0$ (T need not be an integer), and the eigenvalues of A satisfy

$$0 < \lambda_0 < \lambda_1 < \cdots < \lambda_k < \cdots .$$

Let $f(x,t)$ be a Carathéodory function on $\Omega \times \mathbb{R}$. We assume that the function $f(x,t)$ satisfies

$$|f(x,t)| \leq C(|t|+1), \quad x \in \Omega,\ t \in \mathbb{R}. \tag{16.2}$$

We shall prove

Theorem 16.1. *Assume that for some integers $l < m$, the following inequalities hold:*

$$t[f(x,t_1) - f(x,t_0)] \leq at^2, \quad t_j \in \mathbb{R},\ t = t_1 - t_0, \tag{16.3}$$

where $a < \lambda_{m+1}$,

$$a_0 t^2 \leq 2F(x,t) \leq a_1 t^2, \quad |t| < \delta, \tag{16.4}$$

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for some $\delta > 0$, *with* $\lambda_l < a_0 \le a_1 < \lambda_{l+1}$, *where*

$$F(x,t) := \int_0^t f(x,s)ds,$$

and

$$a_2 t^2 - W_1(x) \le 2F(x,t), \quad |t| > K, \tag{16.5}$$

for some $K \ge 0$, *where* $a_2 > \lambda_m$ *and* $W_1 \in L^1(\Omega)$. *Then the equation*

$$Au = f(x,u), \quad u \in D, \tag{16.6}$$

has at least two nontrivial solutions.

Theorem 16.2. *Equation (16.6) will have at least two nontrivial solutions if we assume that for some integers* $l > m$, *the following inequalities hold:*

$$t[f(x,t_1) - f(x,t_0)] \ge at^2, \quad t_j \in \mathbb{R},\ t = t_1 - t_0, \tag{16.7}$$

where $a > \lambda_m$,

$$a_0 t^2 \le 2F(x,t) \le a_1 t^2, \quad |t| < \delta, \tag{16.8}$$

for some $\delta > 0$, *with* $\lambda_l < a_0 \le a_1 < \lambda_{l+1}$,

$$2F(x,t) \le a_2 t^2 + W_2(x), \quad |t| > K, \tag{16.9}$$

for some $K \ge 0$, *where* $a_2 < \lambda_{m+1}$ *and* $W_2 \in L^1(\Omega)$.

More comprehensive theorems will be presented in the next section. Proofs will be given in Section 16.6.

16.3 Statement of the theorems

We use the notation

$$a(u,v) = (Au, v), \quad a(u) = a(u,u), \quad u, v \in D.$$

We define

$$\|u\|_D := \|A^{1/2}u\| \tag{16.10}$$

and

$$G(u) := \|u\|_D^2 - 2\int_\Omega F(x,u)dx.$$

It is known that G is a continuously differentiable functional on the whole of D (cf., e.g., [120]) and

$$(G'(u), v)_D = 2(u,v)_D - 2(f(u), v),$$

where we write $f(u)$ in place of $f(x, u(x))$. In connection with the operator A, the following quantities are very useful. For each fixed positive integer ℓ, we let N_ℓ denote the subspace of D spanned by the eigenfunctions corresponding to $\lambda_0, \dots, \lambda_\ell$, and let $M_\ell = N_\ell^\perp \cap D$. Then $D = M_\ell \oplus N_\ell$. For real a, b, we define

$$
\begin{aligned}
I(u, a, b) &= (Au, u) - a\|u^-\|^2 - b\|u^+\|^2,\\
\gamma_\ell(a) &= \sup\{I(v, a, 0) : v \in N_\ell,\ \|v^+\| = 1\},\\
\Gamma_\ell(a) &= \inf\{I(w, a, 0) : w \in M_\ell,\ \|w^+\| = 1\},\\
F_{1\ell}(w, a, b) &= \sup\{I(v + w, a, b) : v \in N_\ell\},\\
F_{2\ell}(v, a, b) &= \inf\{I(v + w, a, b) : w \in M_\ell\},\\
M_\ell(a, b) &= \inf\{F_{1\ell}(w, a, b) : w \in M_\ell,\ \|w\|_D = 1\},\\
m_\ell(a, b) &= \sup\{F_{2\ell}(v, a, b) : v \in N_\ell,\ \|v\|_D = 1\},\\
\nu_\ell(a) &= \sup\{b : M_\ell(a, b) \geq 0\},\\
\mu_\ell(a) &= \inf\{b : m_\ell(a, b) \leq 0\},
\end{aligned}
$$

where $u^\pm(x) = \max\{\pm u(x), 0\}$. Our first result is

Theorem 16.3. *Assume that for some integers $l < m$, the following inequalities hold:*

(16.11) $$t[f(x, t_1) - f(x, t_0)] \leq a(t^-)^2 + b(t^+)^2, \quad t_j \in \mathbb{R},\ t = t_1 - t_0,$$

where $b < \Gamma_m(a)$,

(16.12) $$a_0(t^-)^2 + b_0(t^+)^2 \leq 2F(x, t) \leq a_1(t^-)^2 + b_1(t^+)^2, \quad |t| < \delta,$$

for some $\delta > 0$, with $a_0, b_0 < \lambda_{l+1}$, $a_1, b_1 > \lambda_l$, $b_0 > \mu_l(a_0)$, and $b_1 < \nu_l(a_1)$,

(16.13) $$a_2(t^-)^2 + b_2(t^+)^2 - W_1(x) \leq 2F(x, t), \quad |t| > K,$$

for some $K \geq 0$, where $a_2, b_2 < \lambda_{m+1}$, $b_2 > \mu_m(a_2)$, and $W_1 \in L^1(\Omega)$. Then (16.6) has at least two nontrivial solutions.

In contrast to this, we have

Theorem 16.4. *Equation (16.6) will have at least two nontrivial solutions if we assume that for some integers $l > m$, the following inequalities hold:*

(16.14) $$t[f(x, t_1) - f(x, t_0)] \geq a(t^-)^2 + b(t^+)^2, \quad t_j \in \mathbb{R},\ t = t_1 - t_0,$$

where $b > \gamma_m(a)$,

(16.15) $$a_0(t^-)^2 + b_0(t^+)^2 \leq 2F(x, t) \leq a_1(t^-)^2 + b_1(t^+)^2, \quad |t| < \delta$$

for some $\delta > 0$, with $a_0, b_0 < \lambda_{l+1}$, $b_0 > \mu_l(a_0)$ and $a_1, b_1 > \lambda_l$, $b_1 < \nu_l(a_1)$,

(16.16) $$2F(x, t) \leq a_2(t^-)^2 + b_2(t^+)^2 + W_2(x), \quad |t| > K,$$

for some $K \geq 0$, where $a_2, b_2 > \lambda_m$, $b_2 < \nu_m(a_2)$, and $W_2 \in L^1(\Omega)$.

It was shown in [114] that the functions $\gamma_l,\ \mu_l,\ \nu_{l-1},\ \Gamma_{l-1}$ all emanate from the point (λ_l, λ_l) and satisfy

$$\Gamma_{l-1}(a) \le \nu_{l-1}(a) \le \mu_l(a) \le \gamma_l(a)$$

on their common domains. It would therefore give a weaker result if we assumed in Theorems 16.3 and 16.4 that $b_0 > \gamma_l(a_0)$ and $b_1 < \Gamma_l(a_1)$. However, the functions $\gamma_l,\ \Gamma_l$ are defined on the whole of $\mathbb{R}$, while the others are not. For cases in which the other functions are not defined, we state the following.

Theorem 16.5. *Theorems 16.3 and 16.4 remain true if we assume that (16.12) holds with $b_0 > \gamma_l(a_0)$, and $b_1 < \Gamma_l(a_1)$ for some $a_0,\ a_1 \in \mathbb{R}$.*

16.4 Some lemmas

The proofs of the theorems of the previous section will be based on a series of lemmas.

Lemma 16.6. *If $b < \Gamma_l(a)$, then there is an $\epsilon > 0$ such that*

$$I(w,a,b) \ge \epsilon \|w\|_D^2, \quad w \in M_l. \tag{16.17}$$

Proof. By the continuity of Γ_l, there is a $t < 1$ such that $b/t < \Gamma_l(a/t)$. Then

$$I(w,a/t,b/t) = \|w\|_D^2 - \frac{a}{t}\|w^-\|^2 - \frac{b}{t}\|w^+\|^2 \ge 0, \quad w \in M_l.$$

Therefore,

$$I(w,a,b) = t\left[\ \|w\|_D^2 - \frac{a}{t}\|w^-\|^2 - \frac{b}{t}\|w^+\|^2\right] + (1-t)\|w\|_D^2 \ge (1-t)\|w\|_D^2.$$

□

Lemma 16.7. *If $b > \gamma_l(a)$, then there is an $\epsilon > 0$ such that*

$$I(v,a,b) \le -\epsilon \|v\|_D^2, \quad v \in N_l. \tag{16.18}$$

Proof. By the continuity of γ_l, there is a $t > 1$ such that $b/t > \gamma_l(a/t)$. Hence,

$$I(v,a/t,b/t) = \|v\|_D^2 - \frac{a}{t}\|v^-\|^2 - \frac{b}{t}\|v^+\|^2 \le 0, \quad v \in N_l,$$

and

$$I(v,a,b) = t\left[\ \|v\|_D^2 - \frac{a}{t}\|v^-\|^2 - \frac{b}{t}\|v^+\|^2\right] + (1-t)\|v\|_D^2 \le (1-t)\|v\|_D^2.$$

□

Lemma 16.8. *If*

$$t[f(x,t_1) - f(x,t_0)] \le a(t^-)^2 + b(t^+)^2, \quad t_j \in \mathbb{R},\ t = t_1 - t_0, \tag{16.19}$$

then

$$(G'(v+w_1) - G'(v+w_0), w) \ge 2I(w,a,b), \quad v, w_j \in D,\ w = w_1 - w_0. \tag{16.20}$$

Proof. We have

$$(f(x, v+w_1) - f(x, v+w_0), w) \le a\|w^-\|^2 + b\|w^+\|^2.$$

Hence,

$$\begin{aligned}(G'(v+w_1) - G'(v+w_0), w)/2 & \\ &= \|w\|_D^2 - (f(x,v+w_1) - f(x,v+w_0), w) \\ &\ge I(w,a,b).\end{aligned}$$

□

Lemma 16.9. *Suppose there are closed subspaces X,Y of E such that $E = X \oplus Y$, and*

$$(G'(x+y_1) - G'(x+y_2), y_1 - y_2) \ge m(\|y_1 - y_2\|), \quad x \in X,\ y_1,\ y_2 \in Y, \tag{16.21}$$

for some function $m(t)$ satisfying
(a)

$$m(t) > 0, \quad t > 0,$$

(b) there is a $t_0 > 0$ such that

$$\int_0^{t_0} \frac{m(t)}{t}\, dt < \infty,$$

(c)

$$\frac{m(t)}{t} \to 0 \quad implies \quad t \to 0,$$

and
(d)

$$\frac{1}{M}\int_0^M \frac{m(t)}{t}\, dt \to \infty \quad as \quad M \to \infty.$$

Then there exists a continuous function $\varphi : X \to Y$ satisfying
(i) $G(x+\varphi(x)) = \min_{y\in Y} G(x+y)$.
(ii) The function $\tilde{G}(x) = G(x+\varphi(x)) : X \to \mathbb{R}$ is of class C^1 and satisfies

$$(\tilde{G}'(x), h) = (G'(x+\varphi(x)), h), \quad h \in X. \tag{16.22}$$

Proof. For each $x \in X$, we define $G_x : Y \to \mathbb{R}$ by $G_x(y) = G(x + y)$. From (16.21), we have

$$(G'_x(y_1) - G'_x(y_2), y_1 - y_2) \geq m(\|y_1 - y_2\|).$$

In particular, $G_x(y)$ is strictly convex (Theorem 13.7). Thus, for each $x \in X$, $G_x(y)$ has a unique critical point $\varphi(x)$. To see this, note that

$$(G'_x(y) - G'_x(0), y) \geq m(\|y\|), \quad y \in Y.$$

Thus,

$$(G'_x(y), y) \geq (G'_x(0), y) + m(\|y\|).$$

Consequently,

$$\begin{aligned}
G_x(y) - G_x(0) &= \int_0^1 \frac{d}{dt} G_x(ty)\, dt \\
&= \int_0^1 \frac{1}{t}(G'_x(ty), ty)\, dt \\
&\geq \int_0^1 \frac{1}{t}[(G'_x(0), ty) + m(\|ty\|)]\, dt \\
&= (G'_x(0), y) + \int_0^1 \frac{1}{t} m(\|ty\|)\, dt \\
&= (G'_x(0), y) + \int_0^{\|y\|} \frac{m(s)}{s}\, ds \\
&\to \infty \text{ as } \|y\| \to \infty.
\end{aligned}$$

Hence, $G_x(y)$ has a unique minimum $\varphi(x)$ (Lemma 13.5). Moreover,

$$G_x(\varphi(x)) = \min_{y \in Y} G_x(y) = \min_{y \in Y} G(x + y).$$

Therefore, $\varphi(x)$ is the only element of Y such that

$$0 = (G'_x(\varphi(x)), y) = (G'(x + \varphi(x)), y), \quad y \in Y. \tag{16.23}$$

Note that φ is continuous. Suppose, on the contrary, that $\{x_n\}$ is a sequence in X such that $x_n \to x \in X$. Let $t_n = \|\varphi(x_n) - \varphi(x)\|$. Since G' is continuous, for each $\varepsilon > 0$ and n sufficiently large, we have

$$|(G'(x_n + \varphi(x)), y)| < \varepsilon\|y\|, \quad y \in Y. \tag{16.24}$$

Because of (16.21), we obtain

$$(G'(x_n + \varphi(x)) - G'(x_n + \varphi(x_n)), \varphi(x) - \varphi(x_n)) \geq m(t_n). \tag{16.25}$$

Since, by (16.23),

$$(G'(x_n + \varphi(x_n)), y) = 0, \quad y \in Y,$$

inequalities (16.24) and (16.25) imply

$$m(t_n)/t_n < \varepsilon.$$

Thus,

$$m(t_n)/t_n \to 0,$$

and consequently, $t_n \to 0$. Hence, φ is continuous. This proves part (i).

For $t > 0$ and $h \in X$, we have

$$\begin{aligned}\frac{\tilde{G}(x+th)-\tilde{G}(x)}{t} &= \frac{G(x+th+\varphi(x+th))-G(x+\varphi(x))}{t}\\ &\le \frac{G(x+th+\varphi(x))-G(x+\varphi(x))}{t}\\ &= \int_0^1 (G'(x+\varphi(x)+sth),h)\,ds.\end{aligned} \tag{16.26}$$

In a similar manner, we see that

$$\begin{aligned}\frac{\tilde{G}(x+th)-\tilde{G}(x)}{t} &= \frac{G(x+th+\varphi(x+th))-G(x+\varphi(x))}{t}\\ &\ge \frac{G(x+th+\varphi(x+th))-G(x+\varphi(x+th))}{t}\\ &\ge \int_0^1 (G'(x+\varphi(x+th)+sth),h)\,ds.\end{aligned}$$

Therefore, since G' and φ are continuous, we have

$$\lim_{t\to 0+}\frac{\tilde{G}(x+th)-\tilde{G}(x)}{t} = (G'(x+\varphi(x)),h).$$

This shows that $\tilde{G}$ has a continuous Gateaux derivative and hence is of class C^1. From the above, we see that

$$(\tilde{G}'(x),h) = (G'(x+\varphi(x)),h), \quad h \in X. \tag{16.27}$$

$\square$

Lemma 16.10. *Let M, N be closed subspaces of a Banach space E such that $\dim N < \infty$ and $E = M \oplus N$. Let $w_0 \neq 0$ be an element of M, and take*

$$\begin{aligned}K_0 &= \{v \in N : \|v\| \le R\} \cup \{sw_0 + v : v \in N, s \ge 0, \|sw_0+v\| = R\},\\ B &= \partial B_\delta \cap M,\ 0 < \delta < R.\end{aligned}$$

Let

$$Q = \{sw_0 + v : v \in N, s \ge 0, \|sw_0+v\| \le R\},$$

Then $K_0 = \partial Q$. *Let* $\xi(u)$ *be a continuous map from* Q *to* E *such that*

$$\xi(v) = v, \quad v \in N \cap B_R$$

and

$$\|\xi(u)\| \geq r > \delta > 0, \quad u \in K_0 \cap \partial B_R.$$

Then

$$\xi(Q) \cap B \neq \phi.$$

Proof. Let P be the projection of E onto N via M. Then the conclusion of the lemma is true iff there is a $u \in Q$ such that

$$S(u) := P\xi(u) + \|\xi(u)\| w_0 = \delta w_0.$$

Let

$$\xi_t(u) = (1-t)\xi(u) + tu$$

and

$$S_t(u) = P\xi_t(u) + [(1-t)\|\xi(u)\| + t\|u\|] w_0.$$

Then $S_0 = S$ and

$$S_1(u) = Pu + \|u\| w_0.$$

Clearly, there is a unique $u \in Q$ such that $S_1(u) = \delta w_0$. The S_t map Q into E, but there is no point on ∂Q that gets mapped into δw_0. If $v \in N \cap B_R$, then $S_t(v) = v + \|v\| w_0$, which cannot equal δw_0. If $u \in \partial Q \backslash N$, then

$$[(1-t)\|\xi(u)\| + t\|u\|] \geq r > \delta.$$

Thus, no such point can be mapped into δw_0. Hence, the Brouwer degree $d(S_t, Q, \delta w_0)$ is defined for $t \in [0, 1]$, and $d(S_0, Q, \delta w_0) = d(S_1, Q, \delta w_0) = 1$. This completes the proof. □

Corollary 16.11. $A = \xi(\partial Q)$ *links* B *[mm].*

Proof. Let $\mathcal{K}$ be the collection of all sets

$$K = \{\eta(Q) : \eta \in C(Q, E),\ \eta(u) = \xi(u), u \in \partial Q\}.$$

It is easily checked that $\mathcal{K}$ is a minimax system. If $\varphi \in \Lambda(A)$, then $\varphi(K) \in \mathcal{K}$ for $K \in \mathcal{K}$. Since $A \cap B = \phi$ and $K \cap B \neq \phi$ for $K \in \mathcal{K}$, we see that A links B [mm]. □

Lemma 16.12. *If* $f(x,t)$ *satisfies (16.19) and* $b < \Gamma_m(a)$, *then there is a continuous map* φ *from* $N_m \to M_m$ *such that*

$$J(v) \equiv G(v + \varphi(v)) = \min_{w \in M_m} G(v + w) \in C^1(N_m, \mathbb{R}), \quad v \in N_m, \tag{16.28}$$

and

$$J'(v) = G'(v + \varphi(v)), \quad v \in N_m. \tag{16.29}$$

Proof. In view of Lemmas 16.6 and 16.8, we have

$$(G'(v+w_1) - G'(v+w_0), w) \geq \epsilon \|w\|_D^2, \quad w \in M_m.$$

We can now apply Lemma 16.9 to arrive at the conclusion. $\square$

Lemma 16.13. *If, in addition,*

$$a_0(t^-)^2 + b_0(t^+)^2 \leq 2F(x,t), \quad |t| < \delta, \tag{16.30}$$

for some $\delta > 0$, *with* $a_0, b_0 < \lambda_{l+1},\ b_0 > \mu_l(a_0),\ l \leq m$, *then there are* $\varepsilon > 0,\ r > 0$ *such that*

$$J(v) \leq -\varepsilon \|v\|_D^2, \quad v \in N_l \cap B_r, \tag{16.31}$$

where

$$B_r = \{u \in D : \|u\|_D \leq r\}.$$

Proof. Let q be any number satisfying

$$2 < q \ \leq 2n/(n-2T),\ 2T \ < n \tag{16.32}$$

$$2 < q \ < \infty,\ n \leq 2T. \tag{16.33}$$

It follows from Theorem 13.6 that there is a continuous map $\tau : N_l \to M_l$ such that

$$\tau(sv) = s\,\tau(v), \quad s \geq 0, \tag{16.34}$$

$$I(v + \tau(v), a_0, b_0) = \inf_{w \in M_l} I(v+w, a_0, b_0), \quad v \in N_l, \tag{16.35}$$

$$\|\tau(v)\|_D \leq C\|v\|_D, \quad v \in N_l. \tag{16.36}$$

Then, for $u = v + \tau(v)$, we have, by (16.2),

$$\begin{aligned} J(v) \leq G(u) &\leq I(u, a_0, b_0) + \int_{|u|>\delta} [a_0(u^-)^2 + b_0(u^+)^2 - 2F(x,u)]\,dx \\ &\leq F_{2l}(v, a_0, b_0) + C\int_{|u|>\delta} |u|^q\,dx \leq m_l(a_0,b_0)\|v\|_D^2 + o(\|v\|_D^2) \leq -\varepsilon\|v\|_D^2 \end{aligned}$$

for r sufficiently small. $\square$

Lemma 16.14. *Assume that*

$$a(t^-)^2 + b(t^+)^2 - W_1(x) \leq 2F(x,t), \quad |t| > K, \tag{16.37}$$

for some $K \geq 0$, *where* $a, b < \lambda_{m+1},\ b \geq \mu_m(a),\ l \leq m$, *and* $W_1 \in L^1(\Omega)$. *Then there is a* $K_1 < \infty$ *such that*

$$J(v) \leq K_1. \tag{16.38}$$

If $b > \mu_m(a)$, *then*

$$J(v) \longrightarrow -\infty \text{ as } \|v\|_D \longrightarrow \infty. \tag{16.39}$$

Proof. For $u = v + w,\ v \in N_m,\ w \in M_m$, we have

$$G(u) \le I(u, a, b) + C \int_{|u|<K} |u|^q \, dx + \int_\Omega W_1(x)\, dx \le I(u, a, b) + K'.$$

Thus,

$$\begin{aligned} J(v) &= \inf_{w \in M_m} G(v + w) \\ &\le \inf_{w \in M_m} I(v + w, a, b) + K' \\ &= F_{2m}(v, a, b) + K' \\ &\le m(a, b)\|v\|_D^2 + K'. \end{aligned}$$

If $b \ge \mu_m(a)$, then $m(a, b) \le 0$. This proves (16.38). If $b > \mu_m(a)$, then $m(a, b) < 0$. This proves (16.39). □

Lemma 16.15. *If $l < m$ and $\lambda_l < a, b < \lambda_{m+1}$, then there are continuous functions $\xi : N_m \cap M_l \to N_l,\ \eta : N_m \cap M_l \to M_m$ homogeneous of degree 1 and such that*
(16.40)
$$I(\xi(y)+\eta(y)+y, a, b) = \sup_{v \in N_l} \inf_{w \in M_m} I(v+w+y, a, b) = \inf_{w \in M_m} \sup_{v \in N_l} I(v+w+y, a, b)$$

for $y \in N_m \cap M_l$.

Proof. Let
$$L_y(v, w) = I(v + w + y, a, b).$$

Then L_y is a strictly convex, lower semicontinuous functional in $w \in M_m$, and strictly concave and continuous in $v \in N_l$. By Theorem 13.6 and Corollary 13.12, for each $y_0 \in N_m \cap M_l$, there are unique elements $v_0 = \xi(y_0) \in N_l,\ w_0 = \eta(y_0) \in M_m$ such that (16.40) holds, i.e., that

$$L_{y_0}(v, w_0) \le L_{y_0}(v_0, w_0) \le L_{y_0}(v_0, w), \quad v \in N_l,\ w \in M_m.$$

The functions ξ, η are clearly homogeneous of degree 1. To prove continuity, let $y_j \to y_0$ in $N_l \cap M_m$, and let $v_j = \xi(y_j),\ w_j = \eta(y_j)$. We note that the functions $v_j,\ w_j$ are bounded in D. Otherwise, it is easy to show that

$$I(v + w_j + y_j, a, b) \longrightarrow \infty \text{ as } j \longrightarrow \infty$$

for any $v \in N_l$, and

$$I(v_j + w + y_j, a, b) \longrightarrow -\infty \text{ as } j \longrightarrow \infty$$

for any $w \in M_m$. This would contradict (16.40). Thus there are renamed subsequences such that $v_j \to v_1,\ w_j \rightharpoonup w_1$ in D. Since

$$\begin{aligned} &I(v + w_j + y_j, a, b) \\ &\qquad \le I(v_j + w_j + y_j, a, b) \\ &\qquad \le I(v_j + w + y_j, a, b), \quad v \in N_l,\ w \in M_m, \end{aligned}$$

we have in the limit

$$\begin{aligned} I(v + w_1 + y_0, a, b) & \\ & \leq I(v_1 + w_1 + y_0, a, b) \\ & \leq I(v_1 + w + y_0, a, b), \quad v \in N_l,\ w \in M_m, \end{aligned}$$

showing that $v_1 = v_0$, $w_1 = w_0$. Since this is true for any subsequence, the result follows. ☐

Lemma 16.16. *If*

$$2F(x,t) \leq a_1(t^-)^2 + b_1(t^+)^2, \quad |t| \leq \delta, \tag{16.41}$$

for some $\delta > 0$, *with* $a_1, b_1 > \lambda_l$, $b_1 < \nu_l(a_1)$, $l < m$, *then there are* $\varepsilon > 0$, $r > 0$ *such that*

$$J(y + \xi(y)) \geq \varepsilon \|y\|_D^2, \quad y \in N_m \cap M_l \cap B_r. \tag{16.42}$$

Proof. By Lemma 16.15

$$\inf_{w \in M_m} I(\xi(y) + y + w, a_1, b_1) = \inf_{w \in M_m} \sup_{v \in N_l} I(v + y + w, a_1, b_1) \tag{16.43}$$

for $y \in N_m \cap M_l$. Then, for $y \in (N_m \cap M_l \cap B_r)\backslash\{0\}$,

$$\begin{aligned} J(\xi(y) + y) &= G(\xi(y) + y + \varphi(\xi(y) + y)) \\ &\geq I(\xi(y) + y + \varphi(\xi(y) + y), a_1, b_1) - o(\|y\|_D^2) \\ &\geq \inf_{w \in M_m} I(\xi(y) + y + w, a_1, b_1) - o(\|y\|_D^2) \\ &= \inf_{w \in M_m} \sup_{v \in N_l} I(v + y + w, a_1, b_1) - o(\|y\|_D^2) \\ &\geq \inf_{w \in M_m} M_l(a,b)\|y + w\|_D^2 - o(\|y\|_D^2) \\ &= M_l(a,b)\|y\|_D^2 - o(\|y\|_D^2) \\ &\geq \varepsilon \|y\|_D^2. \end{aligned}$$

☐

Lemma 16.17. *Assume*

$$t[f(x,t_1) - f(x,t_0)] \geq a(t^-)^2 + b(t^+)^2, \quad t_j \in \mathbb{R},\ t = t_1 - t_0. \tag{16.44}$$

Then

$$(G'(v_1 + w) - G'(v_0 + w), v) \leq 2I(v, a, b), \quad v_j, w \in D,\ v = v_1 - v_0. \tag{16.45}$$

Proof. We have

$$(f(x, v_1 + w) - f(x, v_0 + w), v) \geq a\|v^-\|^2 + b\|v^+\|^2.$$

Hence,

$$\begin{aligned}(G'(v_1 + w) - G'(v_0{+}w), v)/2 \\ &= \|v\|_D^2 - (f(x, v_1 + w) - f(x, v_0 + w), v) \\ &\leq I(v, a, b).\end{aligned}$$

□

Lemma 16.18. *If $f(x,t)$ satisfies (16.44), and $b > \gamma_m(a)$, then there is a continuous map ψ from $M_m \to N_m$ such that*

(16.46) $$J(w) \equiv G(w + \psi(w)) = \max_{v \in N_m} G(v + w) \in C^1(M_m, \mathbb{R}), \quad w \in M_m,$$

and

(16.47) $$J'(w) = G'(w + \psi(w)), \quad w \in M_m.$$

Proof. In view of Lemmas 16.7 and 16.17, we have

$$(G'(v_1 + w) - G'(v_0 + w), v) \leq -\epsilon\|v\|_D^2, \quad v \in N_m.$$

We can now apply Lemma 16.9 to obtain the conclusion. □

Lemma 16.19. *If, in addition,*

(16.48) $$a_0(t^-)^2 + b_0(t^+)^2 \leq 2F(x,t), \quad |t| < \delta,$$

for some $\delta > 0$, with $a_0, b_0 < \lambda_{l+1}$, $b_0 > \mu_l(a_0)$, $l > m$, then there are $\varepsilon > 0$, $r > 0$ such that

(16.49) $$J(y + \eta(y)) \leq -\varepsilon\|y\|_D^2, \quad y \in N_l \cap M_m \cap B_r.$$

Proof. For $y \in M_m \cap N_l$, let $u = y + \eta(y) \in M_m$. By (16.2),

$$\begin{aligned}J(u) &= G(u + \psi(u)) \leq I(u + \psi(u), a_0, b_0) + o(\|u\|_D^2) \\ &\leq \sup_{v \in N_m} I(u + v, a_0, b_0) + o(\|u\|_D^2) \\ &= I(y + \eta(y) + \xi(y), a_0, b_0) + o(\|u\|_D^2) \\ &= \sup_{v \in N_m} \inf_{w \in M_l} I(y + v + w, a_0, b_0) + o(\|u\|_D^2) \\ &= \sup_{v \in N_m} F_{2l}(y + v, a_0, b_0) + o(\|u\|_D^2) \\ &\leq \sup_{v \in N_m} m_l(a_0, b_0)\|y + v\|_D^2 + o(\|u\|_D^2) \\ &\leq -\varepsilon\|y\|_D^2\end{aligned}$$

for r sufficiently small. □

Lemma 16.20. *If*

$$2F(x,t) \le a_1(t^-)^2 + b_1(t^+)^2, \quad |t| \le \delta, \tag{16.50}$$

for some $\delta > 0$, *with* $a_1, b_1 > \lambda_l,\ b_1 < \nu_l(a_1),\ l > m$, *then there are* $\varepsilon > 0,\ r > 0$ *such that*

$$J(w) \ge \varepsilon \|w\|_D^2, \quad w \in M_l \cap B_r. \tag{16.51}$$

Proof. We recall from Theorem 13.6 that there is a continuous map $\theta : M_l \to N_l$ such that

$$\theta(s\,w) = s\,\theta(w), \quad s \ge 0, \tag{16.52}$$

$$I(\theta(w) + w, a_1, b_1) = \sup_{v \in N_l} I(v + w, a_1, b_1), \quad w \in M_l. \tag{16.53}$$

Thus,

$$\begin{aligned} J(w) &\ge G(w + \theta(w), a_1, b_1) \\ &\ge I(w + \theta(w), a_1, b_1) - o(\|w\|_D^2) \\ &= \sup_{v \in N_l} I(v + w, a_1, b_1) - o(\|w\|_D^2) \\ &= F_{1l}(w, a_1, b_1) - o(\|w\|_D^2) \\ &\ge M_l(a_1, b_1)\|w\|_D^2 - o(\|w\|_D^2) \\ &\ge \varepsilon \|w\|_D^2 \end{aligned}$$

for r sufficiently small. □

Lemma 16.21. *Assume that*

$$2F(x,t) \le a(t^-)^2 + b(t^+)^2 + W_1(x), \quad |t| > K, \tag{16.54}$$

for some $K \ge 0$, *where* $a, b > \lambda_m,\ b \le \nu_m(a),\ l \ge m$, *and* $W_1 \in L^1(\Omega)$. *Then there is a* $K_1 < \infty$ *such that*

$$J(w) \ge -K_1, \quad w \in M_m. \tag{16.55}$$

If $b < \nu_m(a)$, *then*

$$J(w) \longrightarrow \infty \text{ as } \|w\|_D \longrightarrow \infty. \tag{16.56}$$

Proof. For $u = v + w,\ v \in N_m,\ w \in M_m$, we have

$$G(u) \ge I(u, a, b) - C\int_{|u|<K} |u|^q\,dx - \int_\Omega W_1(x)dx \ge I(u, a, b) - K'.$$

Thus,

$$\begin{aligned} J(w) = &\sup_{v\in N_m} G(v+w) \\ &\geq \sup_{v\in N_m} I(v+w,a,b) - K' \\ &= F_{1m}(w,a,b) - K' \\ &\geq M_m(a,b)\|w\|_D^2 - K'. \end{aligned}$$

If $b \leq \nu_m(a)$, then $M_m(a,b) \geq 0$. This proves (16.55). If $b < \nu_m(a)$, then $M_m(a,b) > 0$. This proves (16.56). □

Lemma 16.22. *If*

(16.57) $$a_0(t^-)^2 + b_0(t^+)^2 \leq 2F(x,t), \quad |t| < \delta,$$

for some $\delta > 0$, *with* $b_0 > \gamma_l(a_0),\ l \leq m$, *then there are* $\varepsilon > 0,\ r > 0$ *such that*

(16.58) $$J(v) \leq -\varepsilon\|v\|_D^2, \quad v \in N_l \cap B_r,$$

where

$$B_r = \{u \in D : \|u\|_D \leq r\}.$$

Proof. Let q be any number satisfying

$$\begin{aligned} &2 < q \ \leq 2n/(n-2T),\ 2T \ < n \\ &2 < q \ < \infty,\ n \leq 2T. \end{aligned}$$

By (16.2),

$$\begin{aligned} J(v) \leq G(v) \leq I(v,a_0,b_0) &+ \int_{|v|>\delta} [a_0(v^-)^2 + b_0(v^+)^2 - 2F(x,v)]\,dx \\ &\leq -\epsilon\|v\|_D^2 + C\int_{|v|>\delta} |v|^q\ dx \\ &\leq -\epsilon\|v\|_D^2 + o(\|v\|_D^2) \\ &\leq -\varepsilon\|v\|_D^2 \end{aligned}$$

for r sufficiently small. □

Lemma 16.23. *If*

(16.59) $$2F(x,t) \leq a_1(t^-)^2 + b_1(t^+)^2, \quad |t| \leq \delta,$$

for some $\delta > 0$, *with* $b_1 < \Gamma_l(a_1),\ l < m$, *then there are* $\varepsilon > 0,\ r > 0$ *such that*

(16.60) $$J(v) \geq \varepsilon\|v\|_D^2, \quad v \in N_m \cap M_l \cap B_r.$$

Proof. Let $u = v + \varphi(v) \in M_l$. Then

$$\begin{aligned} J(v) = G(u) &\geq I(u, a_1, b_1) + \int_{|u|>\delta} [a_0(u^-)^2 + b_0(u^+)^2 - 2F(x, u)]\, dx \\ &\geq \epsilon \|u\|_D^2 - C \int_{|u|>\delta} |u|^q\, dx \\ &\geq \epsilon \|u\|_D^2 - o(\|u\|_D^2) \\ &\geq \epsilon \|v\|_D^2 - o(\|v\|_D^2) \\ &\geq \varepsilon \|v\|_D^2 \end{aligned}$$

for r sufficiently small, since

$$\|v\|_D \leq \|u\|_D \leq C\|v\|_D.$$

□

Lemma 16.24. *If*

(16.61) $$a_0(t^-)^2 + b_0(t^+)^2 \leq 2F(x, t), \quad |t| < \delta,$$

for some $\delta > 0$, *with* $b_0 > \gamma_l(a_0),\ l \geq m$, *then there are* $\varepsilon > 0,\ r > 0$ *such that*

(16.62) $$J(w) \leq -\varepsilon \|w\|_D^2, \quad w \in N_l \cap M_m \cap B_r.$$

Proof. For $w \in M_m \cap N_l$, let $u = w + \psi(w) \in N_l$. By (16.2),

$$\begin{aligned} J(w) &= G(w + \psi(w)) = G(u) \\ &\leq I(u, a_0, b_0) + \int_{|u|>\delta} [a_0(v^-)^2 + b_0(u^+)^2 - 2F(x, u)]\, dx \\ &\leq -\epsilon \|u\|_D^2 + C \int_{|u|>\delta} |u|^q\, dx \\ &\leq -\epsilon \|u\|_D^2 + o(\|u\|_D^2) \\ &\leq -\varepsilon \|u\|_D^2 \end{aligned}$$

for r sufficiently small. Since

$$\|w\|_D \leq \|u\|_D \leq C\|w\|_D,$$

the result follows. □

Lemma 16.25. *If*

(16.63) $$2F(x, t) \leq a_1(t^-)^2 + b_1(t^+)^2, \quad |t| \leq \delta$$

for some $\delta > 0$, *with* $b_1 < \Gamma_l(a_1),\ l > m$, *then there are* $\varepsilon > 0,\ r > 0$ *such that*

(16.64) $$J(w) \geq \varepsilon \|w\|_D^2, \quad w \in M_l \cap B_r.$$

Proof. We have

$$\begin{aligned} G(w) &\geq I(w, a_1, b_1) + \int_{|w|>\delta} [a_0(u^-)^2 + b_0(w^+)^2 - 2F(x, w)]\, dx \\ &\geq \epsilon \|w\|_D^2 - C \int_{|w|>\delta} |w|^q\, dx \\ &\geq \epsilon \|w\|_D^2 - o(\|w\|_D^2) \\ &\geq \epsilon \|w\|_D^2 - o(\|w\|_D^2) \\ &\geq \varepsilon \|w\|_D^2 \end{aligned}$$

for r sufficiently small. Since

$$J(w) = \sup_{v \in N_l} G(v + w) \geq G(w),$$

the result follows. $\square$

16.5 Local linking

The following theorem will also be used in the proofs of the theorems of Section 16.3. It is also of interest in its own right.

Theorem 16.26. *Let M, N be closed subspaces of a Hilbert space E such that $0 < \dim N < \infty$ and $M = N^\perp$. Let $G \in C^1(E, \mathbb{R})$ satisfy the PS condition and*

$$G(v) \leq 0, \quad v \in N \cap B_R,$$

$$G(w) \geq 0, \quad w \in M \cap B_R,$$

for some $R > 0$. Assume that

$$-\infty < \alpha = \inf_E G < 0.$$

Then G has at least three critical points.

Proof. Since G satisfies the PS condition, it has a minimum point satisfying $G(u_0) = \alpha$. Clearly, 0 is also a critical point. Assume that there are no others. Then the set $\hat{E} = \{u \in E : G'(u) \neq 0\}$ contains all points except u_0 and 0. If $\theta < 1$, then there is a mapping $Y(u)$ from $\hat{E}$ to E that is locally Lipschitz continuous and satisfies

$$\|Y(u)\| \leq 1, \quad \theta \|G'(u)\| \leq (G'(u), Y(u)), \quad u \in \hat{E}.$$

For $v \in N \cap B_R \backslash \{0\}$, let $\sigma(t)v$ be the solution of

$$\sigma'(t) = -Y(\sigma(t)), \quad t \geq 0,\ \sigma(0) = v.$$

Then

$$dG(\sigma(t)v)/dt = (G'(\sigma), \sigma') = -(G'(\sigma), Y(\sigma)) \leq -\theta\|G'(\sigma)\| < 0$$

as long as $\sigma(t)v$ is in $\hat{E}$. Note that $\sigma(t)v$ is continuous in t and v for $v \neq 0$. For each $v \in N \cap B_R \backslash \{0\}$, there is a maximal interval $0 < t < T_v$ in which $\sigma(t)v$ exists and satisfies $G'(\sigma(t)v) \neq 0$ and $G(\sigma(t)v) < 0$. I claim that

$$\sigma(t)v \to u_0 \quad as\ t \to T_v. \tag{16.65}$$

To see this, suppose that $t_k \to T_v$. Then

$$\|\sigma(t_k)v - \sigma(t_j)v\| \leq \left|\int_{t_j}^{t_k} \|Y(\sigma(t)v)\|\, dt\right| \leq |t_k - t_j| \to 0.$$

Thus,

$$\sigma(t_k)v \to h$$

in E. By continuity,

$$G'(\sigma(t_k)v) \to G'(h).$$

If $G'(h) \neq 0$, the solution can be continued beyond T_v, contrary to the way it was chosen. Thus, $G'(h) = 0$, showing that $h = u_0$. Consequently, $\sigma(t_k)v \to u_0$. Since this is true for any such sequence, (16.65) holds. Note that T_v is continuous in v for $v \neq 0$.

Define

$$\begin{aligned} \hat{\sigma}(t)v &= \sigma(t)v, \quad t < T_v, \\ &= u_0, \quad t \geq T_v. \end{aligned} \tag{16.66}$$

Let w_0 be an element of M with unit norm, and take

$$K_0 = \{sw_0 + v : v \in N, s \geq 0, \|sw_0 + v\| = R\}.$$

Let

$$Q = \{sw_0 + v : v \in N, s \geq 0, \|sw_0 + v\| \leq R\}.$$

Let $\varepsilon > 0$ be given, and let $T > 0$ satisfy

$$\frac{T_v^2}{T^2} + \frac{\varepsilon^2}{R^2} \leq 1, \quad v \in N,\ \|v\| = \varepsilon. \tag{16.67}$$

Let $\xi(u)$ be the continuous map from ∂Q to E such that

$$\xi(v) = v, \quad v \in N \cap B_R,$$

and for $u = sw_0 + v \in K_0$,

$$\begin{aligned} \xi(sw_0 + v) &= \hat{\sigma}(Ts/R)v, \quad \|v\| \geq \varepsilon, \\ &= u_0, \quad \|v\| < \varepsilon. \end{aligned} \tag{16.68}$$

By (16.67) and (16.68),

$$\hat{\sigma}(Ts/R)v = u_0, \quad \|v\| = \varepsilon.$$

Hence, ξ is continuous on ∂Q. Moreover,

$$G(\xi(u)) \le 0, \quad u \in \partial Q.$$

In addition,

$$\|\xi(u)\| \ge r > 0, \quad u \in K_0.$$

Let

$$B = \partial B_\delta \cap M, \quad 0 < \delta < r < R.$$

By Corollary 16.11, $A = \xi(\partial Q)$ links B [mm]. Since

$$a_0 := \sup_A G \le b_0 := \inf_B G, \tag{16.69}$$

we can apply Theorem 2.12 to conclude that (1.4) holds. If $a > 0$, this provides a third critical point by the PS condition. If $a = 0$, then there is a sequence satisfying (1.4) and

$$d(u_k, B) \to 0, \quad k \to \infty. \tag{16.70}$$

Since G satisfies the PS condition, there is a subsequence converging to a critical point on B. Again, this provides a third critical point. $\square$

16.6 The proofs

We prove the theorems of Section 16.3. First, we prove Theorem 16.3.

Proof. By Lemma 16.12, it suffices to show that $J(v)$ has two nontrivial solutions. Now J is bounded from above by Lemma 16.14 and satisfies (PS) by (16.39). Moreover,

$$J(v) < 0, \quad v \in N_l \cap B_r \backslash \{0\}, \tag{16.71}$$

by Lemma 16.13, and

$$J(\xi(y) + y) > 0, \quad y \in N_m \cap M_l \cap B_r \backslash \{0\}, \tag{16.72}$$

by Lemma 16.16. Thus, J has a positive maximum on N_m. We can now apply Theorem 16.26 and Lemma 16.9 to obtain the desired conclusion. $\square$

Similarly, we prove Theorem 16.4.

Proof. By Lemma 16.18, it suffices to show that $J(w)$ given by (16.46) has two nontrivial solutions. Now J is bounded from below by Lemma 16.21 and satisfies (PS) by (16.56). Moreover,

$$J(w + \eta(w)) < 0, \quad w \in N_l \cap M_m \cap B_r \backslash \{0\}, \tag{16.73}$$

by Lemma 16.19, and

$$J(w) > 0, \quad w \in M_l \cap B_r \backslash \{0\} \tag{16.74}$$

by Lemma 16.20. Thus, J has a negative minimum on M_m. We can now apply Theorem 16.26 and Lemma 16.9 to obtain the desired conclusion. □

Next, we prove Theorem 16.5.

Proof. With reference to Theorem 16.3, we note that, by Lemma 16.12, it suffices to show that $J(v)$ has two nontrivial solutions. Now J is bounded from above by Lemma 16.14 and satisfies (PS) by (16.39). Moreover,

$$J(v) < 0, \quad v \in N_l \cap B_r \backslash \{0\} \tag{16.75}$$

by Lemma 16.22, and

$$J(v) > 0, \quad v \in N_m \cap M_l \cap B_r \backslash \{0\} \tag{16.76}$$

by Lemma 16.23. Thus J has a positive maximum on N_m. We can now apply Theorem 16.26 and Lemma 16.9 to obtain the desired conclusion. With respect to Theorem 16.4, we note that by Lemma 16.18, it suffices to show that $J(w)$ given by (16.46) has two nontrivial solutions. Now J is bounded from below by Lemma 16.21 and satisfies (PS) by (16.56). Moreover,

$$J(w) < 0, \quad w \in N_l \cap M_m \cap B_r \backslash \{0\}, \tag{16.77}$$

by Lemma 16.24, and

$$J(w) > 0, \quad w \in M_l \cap B_r \backslash \{0\}, \tag{16.78}$$

by Lemma 16.25. Thus, J has a negative minimum on M_m. We can now apply Theorem 16.26 and Lemma 16.9 to obtain the desired conclusion. □

16.7 Notes and remarks

In his studies of semilinear elliptic problems with jumping nonlinearities, Các [29]–[34] proved the following.

Theorem 16.27. *Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geq 2$, with smooth boundary $\partial\Omega$. Let $0 < \lambda_0 < \lambda_1 < \cdots < \lambda_k < \cdots$ be the sequence of distinct eigenvalues of the eigenvalue problem*

$$-\Delta u = \lambda u \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega. \tag{16.79}$$

Let $p(t)$ be a continuous function such that $p(0) = 0$ and

$$p(t)/t \longrightarrow a \quad \text{as} \quad t \longrightarrow -\infty$$

and

$$p(t)/t \longrightarrow b \quad \text{as} \quad t \longrightarrow +\infty.$$

Assume that for some $k \geq 1$, we have $a \in (\lambda_{k-1}, \lambda_k)$, $b \in (\lambda_k, \lambda_{k+1})$, and the only solution of

$$-\Delta u = bu^+ - au^- \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega \tag{16.80}$$

is $u \equiv 0$, where $u^\pm = \max[\pm u, 0]$. Assume further that

$$\frac{p(s) - p(t)}{s - t} \leq \nu < \lambda_{k+1}, \quad s, t \in \mathbb{R}, \ s \neq t. \tag{16.81}$$

Assume also that $p'(0)$ exists and satisfies $p'(0) \in (\lambda_{j-1}, \lambda_j)$ for some $j \leq k$. Then

$$-\Delta u = p(u) \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega \tag{16.82}$$

has at least two nontrivial solutions.

This theorem generalizes the work of Gallouët and Kavian [72], [73] which required λ_k to be a simple eigenvalue and the left-hand side of (16.81) to be sandwiched in between λ_{k-1} and λ_{k+1} and bounded away from both of them. Các proves a counterpart of the theorem in which the inequalities are reversed.

In the present chapter we generalized this theorem and its reverse-inequality counterpart by not requiring $p(t)/t$ to converge to limits at either $\pm\infty$ or ± 0. Rather, we worked with the primitive

$$F(x, t) := \int_0^t f(x, s) ds$$

and bounded $2F(x, t)/t^2$ near $\pm\infty$ and ± 0 [we replaced $p(t)$ with a function $f(x, t)$ depending on x as well]. Our main assumptions were

$$t[f(x, t_1) - f(x, t_0)] \leq a(t^-)^2 + b(t^+)^2, \quad t_j \in \mathbb{R}, \ t = t_1 - t_0, \tag{16.83}$$

$$a_0(t^-)^2 + b_0(t^+)^2 \leq 2F(x, t) \leq a_1(t^-)^2 + b_1(t^+)^2, \quad |t| < \delta, \tag{16.84}$$

for some $\delta > 0$,

$$a_2(t^-)^2 + b_2(t^+)^2 - W_1(x) \leq 2F(x, t), \quad |t| > K, \tag{16.85}$$

for some $K > 0$ and $W_1 \in L^1(\Omega)$, where the constants $a, a_0, a_1, a_2, b, b_0, b_1, b_2$ are suitably chosen (they include the cases considered by Các). The advantage of such inequalities is that they do not restrict the expression $2F(x,t)/t^2$ or $f(x,t)/t$ to any particular interval.

The results of this chapter come from [101] with changes in the proofs. Theorem 16.26 is from [28] with variations made in the proof. Lemma 16.9 is due to Castro [38].

Chapter 17

Second-Order Periodic Systems

17.1 Introduction

In this chapter we study a general system of second-order differential equations, and we look for periodic solutions. We show that for several sets of hypotheses such systems can be solved by the methods used in the book.

We consider the following problem. One wishes to solve

$$-\ddot{x}(t) = \nabla_x V(t, x(t)), \tag{17.1}$$

where

$$x(t) = (x_1(t), \cdots, x_n(t)) \tag{17.2}$$

is a map from $I = [0, T]$ to $\mathbb{R}^n$ such that each component $x_j(t)$ is a periodic function with period T, and the function $V(t, x) = V(t, x_1, \ldots, x_n)$ is continuous from $\mathbb{R}^{n+1}$ to $\mathbb{R}$ with

$$\nabla_x V(t, x) = (\partial V/\partial x_1, \ldots, \partial V/\partial x_n) \in C(\mathbb{R}^{n+1}, \mathbb{R}^n). \tag{17.3}$$

For each $x \in \mathbb{R}^n$, the function $V(t, x)$ is periodic in t with period T.

We shall study this problem under the following assumptions:

1. $$V(t, x) \geq 0, \quad t \in I, \ x \in \mathbb{R}^n.$$

2. There are constants $m > 0$, $\alpha \leq 6m^2/T^2$ such that
$$V(t, x) \leq \alpha, \quad |x| \leq m, \ t \in I, \ x \in \mathbb{R}^n.$$

3. There is a constant $\mu > 2$ such that
$$\frac{H_\mu(t, x)}{|x|^2} \leq W(t) \in L^1(I), \quad |x| \geq C, \ t \in I, \ x \in \mathbb{R}^n, \tag{17.4}$$

M. Schechter, *Minimax Systems and Critical Point Theory*, DOI 10.1007/978-0-8176-4902-9_17,

and

$$\limsup_{|x|\to\infty} \frac{H_\mu(t,x)}{|x|^2} \le 0, \tag{17.5}$$

where

$$H_\mu(t,x) = \mu V(t,x) - \nabla_x V(t,x) \cdot x. \tag{17.6}$$

4. There is a subset $e \subset I$ of positive measure such that

$$\liminf_{|x|\to\infty} \frac{V(t,x)}{|x|^2} > 0, \quad t \in e. \tag{17.7}$$

We have

Theorem 17.1. *Under the above hypotheses, system (17.1) has a solution.*

As a variant of Theorem 17.1, we have

Theorem 17.2. *The conclusion in Theorem 17.1 is the same if we replace hypothesis 2 with*

2A. *There is a constant $q > 2$ such that*

$$V(t,x) \le C(|x|^q + 1), \quad t \in I, \ x \in \mathbb{R}^n,$$

and there are constants $m > 0$, $\alpha < 2\pi^2/T^2$ such that

$$V(t,x) \le \alpha|x|^2, \quad |x| \le m, \ t \in I, \ x \in \mathbb{R}^n.$$

We also have

Theorem 17.3. *The conclusion of Theorem 17.1 holds if we replace hypothesis 3 with*

3A. *There is a constant $\mu < 2$ such that*

$$\frac{H_\mu(t,x)}{|x|^2} \ge -W(t) \in L^1(I), \quad |x| \ge C, \ t \in I, \ x \in \mathbb{R}^n, \tag{17.8}$$

and

$$\liminf_{|x|\to\infty} \frac{H_\mu(t,x)}{|x|^2} \ge 0. \tag{17.9}$$

And we have

Theorem 17.4. *The conclusion of Theorem 17.1 holds if we replace hypothesis 1 with*

1A.

$$0 \le V(t,x) \le C(|x|^2 + 1), \quad t \in I, \ x \in \mathbb{R}^n.$$

and hypothesis 3 with

3B. *The function given by*

$$H(t,x) = 2V(t,x) - \nabla_x V(t,x) \cdot x \tag{17.10}$$

satisfies

$$H(t,x) \leq W(t) \in L^1(I), \quad |x| \geq C, \ t \in I, \ x \in \mathbb{R}^n, \tag{17.11}$$

and

$$H(t,x) \to -\infty, \quad |x| \to \infty, \ t \in I, \ x \in \mathbb{R}^n. \tag{17.12}$$

Theorems 17.1–17.4 show the existence of solutions, which conceivably could be constants. The following theorems provide the existence of non-constant solutions.

Theorem 17.5. *If we replace hypothesis 4 in Theorem 17.1 with*

4A. *There are constants $\beta > 2\pi^2/T^2$ and C such that*

$$V(t,x) \geq \beta|x|^2, \quad |x| > C, \ t \in I, \ x \in \mathbb{R}^n,$$

then system (17.1) has a nonconstant solution.

As a variant of Theorem 17.5, we have

Theorem 17.6. *The conclusion in Theorem 17.5 is the same if we replace hypothesis 2 with hypothesis 2A.*

We also have

Theorem 17.7. *The conclusion of Theorem 17.5 holds if we replace hypothesis 1 with hypothesis 1A and hypothesis 3 with hypothesis 3B.*

We shall prove Theorems 17.1–17.7 in the next section. We use the linking method of Chapter 2.

17.2 Proofs of the theorems

We now give the proof of Theorem 17.1.

Proof. Let X be the set of vector functions $x(t)$ described above. It is a Hilbert space with norm satisfying

$$\|x\|_X^2 = \sum_{j=1}^n \|x_j\|_{H^1}^2.$$

We also write

$$\|x\|^2 = \sum_{j=1}^n \|x_j\|^2,$$

where $\|\cdot\|$ is the $L^2(I)$ norm.

Let

$$N = \{x(t) \in X : x_j(t) \equiv \text{ constant}, \quad 1 \le j \le n\},$$

and $M = N^\perp$. The dimension of N is n, and $X = M \oplus N$. The following is known (cf., e.g., Proposition 1.3 of [95]).

Lemma 17.8. *If $x \in M$, then*

$$\|x\|_\infty^2 \le \frac{T}{12}\|\dot{x}\|^2$$

and

$$\|x\| \le \frac{T}{2\pi}\|\dot{x}\|.$$

Proof. It suffices to prove the lemma for continuously differential periodic functions. First, consider the case $T = 2\pi$. Using Fourier series, we have

$$x = \sum_{k=-\infty}^{\infty} \alpha_k \varphi_k, \tag{17.13}$$

where

$$\alpha_k = (x, \bar{\varphi}_k), \quad k = 0, \pm 1, \pm 2, \ldots, \tag{17.14}$$

and

$$\varphi_k(x) = \frac{1}{\sqrt{2\pi}} e^{ikx}, \quad k = 0, \pm 1, \pm 2, \ldots \tag{17.15}$$

Thus,

$$\|x\|^2 = \sum_{k=-\infty}^{\infty} |\alpha_k|^2. \tag{17.16}$$

If $x \perp M$, then $\alpha_0 = 0$, and

$$\|x\|^2 \le \sum_{k=-\infty}^{\infty} |k\alpha_k|^2 = \|\dot{x}\|^2.$$

Moreover,

$$\|x\|_\infty^2 \le \frac{1}{2\pi}\left(\sum_{k=-\infty}^{\infty} |\alpha_k|\right)^2 \le \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} |k\alpha_k|^2 \left(2\sum_{k=1}^{\infty} k^{-2}\right) = \frac{1}{2\pi}\|\dot{x}\|^2\pi^2/3.$$

This proves the lemma for the case $T = 2\pi$. Otherwise, we let $y(t) = x(Tt/2\pi)$. Then

$$\|x\|^2 = \frac{T}{2\pi}\|y\|^2, \quad \|\dot{x}\|^2 = \frac{2\pi}{T}\|\dot{y}\|^2, \quad \|x\|_\infty = \|y\|_\infty.$$

Thus,

$$\|x\|^2 = \frac{T}{2\pi}\|y\|^2 \leq \frac{T}{2\pi}\|\dot{y}\|^2 = \left(\frac{T}{2\pi}\right)^2 \|\dot{x}\|^2$$

and

$$\|x\|_\infty = \|y\|_\infty \leq \frac{\pi}{6}\|\dot{y}\|^2 = \frac{\pi}{6}\frac{T}{2\pi}\|\dot{x}\|^2 = \frac{T}{12}\|\dot{x}\|^2.$$

$\square$

Note that it follows that

$$\|x\|_\infty \leq C\|x\|_X, \quad x \in X.$$

We define

$$G(x) = \|\dot{x}\|^2 - 2\int_I V(t, x(t))\, dt, \quad x \in X. \tag{17.17}$$

For each $x \in X$, write $x = v + w$, where $v \in N,\ w \in M$. For convenience, we shall use the following equivalent norm for X:

$$\|x\|_X^2 = \|\dot{w}\|^2 + \|v\|^2.$$

If $x \in M$ and

$$\|\dot{x}\|^2 = \rho^2 = \frac{12}{T}m^2,$$

then Lemma 17.8 implies that $\|x\|_\infty \leq m$, and we have by hypothesis 2 that $V(t, x) \leq \alpha$. Hence,

$$\begin{aligned} G(x) &\geq \|\dot{x}\|^2 - 2\int_{|x|<m} \alpha\, dt \\ &\geq \rho^2 - 2\alpha T \geq 0. \end{aligned} \tag{17.18}$$

We also note that hypothesis 1 implies

$$G(v) \leq 0, \quad v \in N. \tag{17.19}$$

Take

$$\begin{aligned} A &= \partial B_\rho \cap M, \quad \rho^2 = \frac{12}{T}m^2, \\ B &= N, \end{aligned}$$

where

$$B_\sigma = \{x \in X : \|x\|_X < \sigma\}.$$

By Example 2 of Section 3.4, A links B. Moreover, if R is sufficiently large,

$$\sup_A[-G] \leq 0 \leq \inf_B[-G]. \tag{17.20}$$

Hence, we may apply Theorem 3.4 to conclude that there is a sequence $\{x^{(k)}\} \subset X$ such that

$$G(x^{(k)}) = \|\dot{x}^{(k)}\|^2 - 2\int_I V(t, x^{(k)}(t))\, dt \to c \geq 0, \tag{17.21}$$

$$(G'(x^{(k)}), z)/2 = (\dot{x}^{(k)}, \dot{z}) - \int_I \nabla_x V(t, x^{(k)}) \cdot z(t)\, dt \to 0, \quad z \in X, \tag{17.22}$$

and

$$(G'(x^{(k)}), x^{(k)})/2 = \|\dot{x}^{(k)}\|^2 - \int_I \nabla_x V(t, x^{(k)}) \cdot x^{(k)}\, dt \to 0. \tag{17.23}$$

If

$$\rho_k = \|x^{(k)}\|_X \leq C,$$

then there is a renamed subsequence such that $x^{(k)}$ converges to a limit $x \in X$ weakly in X and uniformly on I. From (17.22), we see that

$$(G'(x), z)/2 = (\dot{x}, \dot{z}) - \int_I \nabla_x V(t, x(t)) \cdot z(t)\, dt = 0, \quad z \in X,$$

from which we conclude easily that x is a solution of (17.1).

If

$$\rho_k = \|x^{(k)}\|_X \to \infty,$$

let $\tilde{x}^{(k)} = x^{(k)}/\rho_k$. Then $\|\tilde{x}^{(k)}\|_X = 1$. Let $\tilde{x}^{(k)} = \tilde{w}^{(k)} + \tilde{v}^{(k)}$, where $\tilde{w}^{(k)} \in M$ and $\tilde{v}^{(k)} \in N$. There is a renamed subsequence such that $\|[\tilde{x}^{(k)}]^{\cdot}\| \to r$ and $\|\tilde{x}^{(k)}\| \to \tau$, where $r^2 + \tau^2 = 1$. From (17.21) and (17.23), we obtain

$$\|[\tilde{x}^{(k)}]^{\cdot}\|^2 - 2\int_I V(t, x^{(k)}(t))\, dt/\rho_k^2 \to 0$$

and

$$\|[\tilde{x}^{(k)}]^{\cdot}\|^2 - \int_I \nabla_x V(t, x^{(k)}) \cdot x^{(k)}\, dt/\rho_k^2 \to 0.$$

Thus,

$$2\int_I V(t, x^{(k)}(t))\, dt/\rho_k^2 \to r^2 \tag{17.24}$$

and

$$\int_I \nabla_x V(t, x^{(k)}) \cdot x^{(k)}\, dt/\rho_k^2 \to r^2. \tag{17.25}$$

Hence,

$$\int_I H_\mu(t, x^{(k)}(t))\, dt/\rho_k^2 \to \left(\frac{\mu}{2} - 1\right) r^2. \tag{17.26}$$

Note that

$$|\tilde{x}^{(k)}(t)| \le C\|\tilde{x}^{(k)}\|_X = C.$$

If

$$|x^{(k)}(t)| \to \infty,$$

then, by hypothesis 3,

$$\limsup \frac{H_\mu(t, x^{(k)}(t))}{\rho_k^2} \le \limsup \frac{H_\mu(t, x^{(k)}(t))}{|x^{(k)}(t)|^2}|\tilde{x}^{(k)}(t)|^2 \le 0.$$

If

$$|x^{(k)}(t)| \le C,$$

then

$$\frac{H_\mu(t, x^{(k)}(t))}{\rho_k^2} \to 0.$$

Hence,

$$\limsup \int_I H_\mu(t, x^{(k)}(t))\, dt/\rho_k^2 \le 0.$$

Thus, by (17.26),

$$\left(\frac{\mu}{2} - 1\right) r^2 \le 0.$$

If $r \neq 0$, this contradicts the fact that $\mu > 2$. If $r = 0$, then $\tilde{w}^{(k)} \to 0$ uniformly in I by Lemma 17.8. Moreover, $T|\tilde{v}^{(k)}|^2 = \|\tilde{v}^{(k)}\|^2 \to 1$. Thus, there is a renamed subsequence such that $\tilde{v}^{(k)} \to \tilde{v}$ in N with $|\tilde{v}|^2 = 1/T$. Hence, $\tilde{x}^{(k)} \to \tilde{v}$ uniformly in I. Consequently, $|x^{(k)}| \to \infty$ uniformly in I. Thus, by hypothesis 4,

$$\liminf \int_I V(t, x^{(k)}(t))\, dt/\rho_k^2 \ge \int_e \liminf \frac{V(t, x^{(k)}(t))}{|x^{(k)}(t)|^2}|\tilde{x}^{(k)}(t)|^2\, dt > 0.$$

This contradicts (17.24). Hence, the ρ_k are bounded, and the proof is complete. □

The proof of Theorem 17.2 is similar to that of Theorem 17.1 with the exception of inequality (17.18) resulting from hypothesis 2. In its place we reason as follows: If $x \in M$, we have, by hypothesis 2A,

$$\begin{aligned} G(x) &\ge \|\dot{x}\|^2 - 2\int_{|x|<m} \alpha |x(t)|^2\, dt - C\int_{|x|>m} (|x|^q + 1)\, dt \\ &\ge \|\dot{x}\|^2 - 2\alpha\|x\|^2 - C(1 + m^{2-q} + m^{-q})\int_{|x|>m} |x|^q\, dt \\ &\ge \|\dot{x}\|^2(1 - [2\alpha T^2/4\pi^2]) - C'\int_{|x|>m} |x|^q\, dt \\ &\ge (1 - [\alpha T^2/2\pi^2])\|x\|_X^2 - C''\int_I \|x\|_X^q\, dt \end{aligned}$$

$$
\begin{aligned}
&\geq (1-[\alpha T^2/2\pi^2])\|x\|_X^2 - C'''\|x\|_X^q \\
&= \left(1-[\alpha T^2/2\pi^2] - C'''\|x\|_X^{q-2}\right)\|x\|_X^2
\end{aligned}
$$

by Lemma 17.8. Hence, we have

Lemma 17.9.

$$G(x) \geq \varepsilon\|x\|_X^2, \quad \|x\|_X \leq \rho, \ x \in M, \tag{17.27}$$

for $\rho > 0$ sufficiently small, where $\varepsilon < 1-[\alpha T^2/2\pi^2]$.

The remainder of the proof is essentially the same.

In proving Theorem 17.3, we follow the proof of Theorem 17.1 until we reach (17.26). Then we reason as follows. If

$$|x^{(k)}(t)| \to \infty,$$

then

$$\liminf \frac{H_\mu(t, x^{(k)}(t))}{\rho_k^2} \geq \liminf \frac{H_\mu(t, x^{(k)}(t))}{|x^{(k)}(t)|^2}|\tilde{x}^{(k)}(t)|^2 \geq 0.$$

If

$$|x^{(k)}(t)| \leq C,$$

then

$$\frac{H_\mu(t, x^{(k)}(t))}{\rho_k^2} \to 0.$$

Hence,

$$\liminf \int_I H_\mu(t, x^{(k)}(t))\, dt/\rho_k^2 \geq 0.$$

Thus, by (17.26),

$$\left(\frac{\mu}{2} - 1\right) r^2 \geq 0.$$

If $r \neq 0$, this contradicts the fact that $\mu < 2$. If $r = 0$, then $\tilde{w}^{(k)} \to 0$ uniformly in I by Lemma 17.8. Moreover, $T|\tilde{v}^{(k)}|^2 = \|\tilde{v}^{(k)}\|^2 \to 1$. Hence, there is a renamed subsequence such that $\tilde{v}^{(k)} \to \tilde{v}$ in N with $|\tilde{v}|^2 = 1/T$. Hence, $\tilde{x}^{(k)} \to \tilde{v}$ uniformly in I. Consequently, $|x^{(k)}(t)| \to \infty$ uniformly in I. Thus, by Hypothesis 4,

$$\liminf \int_I V(t, x^{(k)}(t))\, dt/\rho_k^2 \geq \int_e \liminf \frac{V(t, x^{(k)}(t))}{|x^{(k)}(t)|^2}|\tilde{x}^{(k)}(t)|^2\, dt > 0.$$

This contradicts (17.24). Hence, the ρ_k are bounded, and the proof is complete.

In proving Theorem 17.4, we follow the proof of Theorem 17.1 until (17.26). Assume first that $r > 0$. Note that (17.21) and (17.23) imply that

$$\int_I H(t, x^{(k)}(t))\, dt \to -c. \tag{17.28}$$

On the other hand, by hypothesis 1A, we have

$$\begin{aligned} 0 \leftarrow \|[\tilde{x}^{(k)}]^{\cdot}\|^2 - 2\int_I V(t, x^{(k)}(t))\,dt/\rho_k^2 \\ \geq \|[\tilde{x}^{(k)}]^{\cdot}\|^2 - 2C\int_I (|\tilde{x}^{(k)}(t)|^2 + \rho_k^{-2})\,dt \\ \to r^2 - 2C\int_I |\tilde{x}(t)|^2\,dt. \end{aligned}$$

Thus, $\tilde{x}(t) \not\equiv 0$. Let $\Omega_0 \subset I$ be the set on which $\tilde{x}(t) \neq 0$. The measure of Ω_0 is positive. Moreover, $|x^{(k)}(t)| \to \infty$ as $k \to \infty$ for $t \in \Omega_0$. Hence,

$$\int_I H(t, x^{(k)}(t))\,dt \leq \int_{\Omega_0} H(t, x^{(k)}(t))\,dt + \int_{I\setminus\Omega_0} W(t)\,dt \to -\infty$$

by hypothesis 3A. But this contradicts (17.28). If $r = 0$, then $\tilde{w}^{(k)} \to 0$ uniformly in I by Lemma 17.8. Moreover, $T|\tilde{v}^{(k)}|^2 = \|\tilde{v}^{(k)}\|^2 \to 1$. Thus, there is a renamed subsequence such that $\tilde{v}^{(k)} \to \tilde{v}$ in N with $|\tilde{v}|^2 = 1/T$. Hence, $\tilde{x}^{(k)} \to \tilde{v}$ uniformly in I. Consequently, $|x^{(k)}| \to \infty$ uniformly in I. Thus, by hypothesis 4,

$$\liminf \int_I V(t, x^{(k)}(t))\,dt/\rho_k^2 \geq \int_e \liminf \frac{V(t, x^{(k)}(t))}{|x^{(k)}(t)|^2}|\tilde{x}^{(k)}(t)|^2\,dt > 0.$$

This contradicts (17.24). Hence, the ρ_k are bounded, and the proof is complete.

17.3 Nonconstant solutions

We now turn to the proofs of Theorems 17.5–17.7.

First, we prove Theorem 17.5.

Proof. As before, we define

$$G(x) = \|\dot{x}\|^2 - 2\int_I V(t, x(t))\,dt, \quad x \in X. \tag{17.29}$$

For each $x \in X$, write $x = v + w$, where $v \in N,\ w \in M$. As before, if $x \in M$ and

$$\|\dot{x}\|^2 = \rho^2 = \frac{12}{T}m^2,$$

then Lemma 17.8 and hypothesis 2 imply that

$$\begin{aligned} G(x) &\geq \|\dot{x}\|^2 - 2\int_{|x|<m} \alpha\,dt \\ &\geq \rho^2 - 2\alpha T \geq 0. \end{aligned} \tag{17.30}$$

Note that hypothesis 4A is equivalent to

$$V(t, x) \geq \beta|x|^2 - C, \quad t \in I,\ x \in \mathbb{R}^n, \tag{17.31}$$

for some constant C.

Next, let
$$y(t) = v + sw_0,$$
where $v \in N$, $s \geq 0$, and
$$w_0 = (\sin(2\pi t/T), 0, \dots, 0).$$
Then $w_0 \in M$, and
$$\|w_0\|^2 = T/2, \quad \|\dot{w}_0\|^2 = 2\pi^2/T.$$
Note that
$$\|y\|^2 = \|v\|^2 + s^2T/2 = T|v|^2 + Ts^2/2.$$
Consequently,
$$\begin{aligned} G(y) &= s^2\|\dot{w}_0\|^2 - 2\int_I V(t, y(t))\, dt \\ &\leq 2\pi^2 s^2/T - 2\beta \int_I |y(t)|^2\, dt + TC \\ &\leq 2\pi^2 s^2/T - 2\beta(\|v\|^2 + Ts^2/2) + TC \\ &\leq (2\pi^2 - \beta T^2)s^2/T - 2T\beta|v|^2 + TC \\ &\to -\infty \ \text{ as } \ s^2 + |v|^2 \to \infty. \end{aligned}$$

We also note that hypothesis 1 implies
$$G(v) \leq 0, \quad v \in N. \tag{17.32}$$
Take
$$\begin{aligned} A &= \{v \in N : \|v\| \leq R\} \cup \{sw_0 + v : v \in N, s \geq 0, \|sw_0 + v\| = R\}, \\ B &= \partial B_\rho \cap M, \ 0 < \rho < R, \end{aligned}$$
where
$$B_\sigma = \{x \in X : \|x\|_X < \sigma\}.$$
By Example 3 of Section 3.4, A links B. Moreover, if R is sufficiently large,
$$\sup_A G \leq 0 \leq \inf_B G. \tag{17.33}$$

Hence, we may apply Theorem 3.4 to conclude that there is a sequence $\{x^{(k)}\} \subset X$ such that
$$G(x^{(k)}) = \|\dot{x}^{(k)}\|^2 - 2\int_I V(t, x^{(k)}(t))\, dt \to c \geq 0, \tag{17.34}$$
$$(G'(x^{(k)}), z)/2 = (\dot{x}^{(k)}, \dot{z}) - \int_I \nabla_x V(t, x^{(k)}) \cdot z(t)\, dt \to 0, \quad z \in X, \tag{17.35}$$

and

$$(17.36)\qquad (G'(x^{(k)}), x^{(k)})/2 = \|\dot{x}^{(k)}\|^2 - \int_I \nabla_x V(t, x^{(k)}) \cdot x^{(k)}\, dt \to 0.$$

If

$$\rho_k = \|x^{(k)}\|_X \le C,$$

then there is a renamed subsequence such that $x^{(k)}$ converges to a limit $x \in X$ weakly in X and uniformly on I. From (17.35), we see that

$$(G'(x), z)/2 = (\dot{x}, \dot{z}) - \int_I \nabla_x V(t, x(t)) \cdot z(t)\, dt = 0, \quad z \in X,$$

from which we conclude easily that x is a solution of (17.1). By (17.34), we see that

$$G(x) \ge c \ge 0,$$

showing that $x(t)$ is not a constant, for if $c > 0$ and $x \in N$, then

$$G(x) = -2 \int_I V(t, x(t))\, dt \le 0.$$

If $c = 0$, we know that $d(x^{(k)}, B) \to 0$ by Theorem 2.12. Hence, there is a sequence $\{y^{(k)}\} \subset B$ such that $x^{(k)} - y^{(k)} \to 0$ in X. If $v \in N$, then

$$(x, v) = (x - x^{(k)}, v) + (x^{(k)} - y^{(k)}, v) \to 0$$

since $y^{(k)} \in M$. Thus, $x \in M$.

If

$$\rho_k = \|x^{(k)}\|_X \to \infty,$$

let $\tilde{x}^{(k)} = x^{(k)}/\rho_k$. Then $\|\tilde{x}^{(k)}\|_X = 1$. Let $\tilde{x}^{(k)} = \tilde{w}^{(k)} + \tilde{v}^{(k)}$, where $\tilde{w}^{(k)} \in M$ and $\tilde{v}^{(k)} \in N$. There is a renamed subsequence such that $\|[\tilde{w}^{(k)}]^{\cdot}\| \to r$ and $\|\tilde{v}^{(k)}\| \to \tau$, where $r^2 + \tau^2 = 1$. From (17.21) and (17.23) we obtain

$$\|[\tilde{x}^{(k)}]^{\cdot}\|^2 - 2 \int_I V(t, x^{(k)}(t))\, dt/\rho_k^2 \to 0$$

and

$$\|[\tilde{x}^{(k)}]^{\cdot}\|^2 - \int_I \nabla_x V(t, x^{(k)}) \cdot x^{(k)}\, dt/\rho_k^2 \to 0.$$

Thus,

$$(17.37)\qquad 2 \int_I V(t, x^{(k)}(t))\, dt/\rho_k^2 \to r^2$$

and

$$(17.38)\qquad \int_I \nabla_x V(t, x^{(k)}) \cdot x^{(k)}\, dt/\rho_k^2 \to r^2.$$

Hence,

$$\int_I H_\mu(t, x^{(k)}(t))\, dt/\rho_k^2 \to \left(\frac{\mu}{2} - 1\right) r^2. \tag{17.39}$$

By hypothesis 3, the left-hand side of (17.37) is

$$\geq (2\beta \|x^{(k)}\|^2 - 4\pi C)/\rho_k^2 \to 2\beta\tau^2.$$

Hence,

$$r^2 \geq 2\beta\tau^2 = 2\beta(1 - r^2),$$

showing that

$$r^2 \geq \frac{2\beta}{1 + 2\beta} > 0.$$

Note that

$$|\tilde{x}^{(k)}(t)| \leq C\|\tilde{x}^{(k)}\|_X = C.$$

If

$$|x^{(k)}(t)| \to \infty,$$

then

$$\limsup \frac{H_\mu(t, x^{(k)}(t))}{\rho_k^2} \leq \limsup \frac{H_\mu(t, x^{(k)}(t))}{|x^{(k)}(t)|^2} |\tilde{x}^{(k)}(t)|^2 \leq 0.$$

If

$$|x^{(k)}(t)| \leq C,$$

then

$$\frac{H_\mu(t, x^{(k)}(t))}{\rho_k^2} \to 0.$$

Hence,

$$\limsup \int_I H_\mu(t, x^{(k)}(t))\, dt/\rho_k^2 \leq 0.$$

In view of (17.39), this implies that

$$\left(\frac{\mu}{2} - 1\right) r^2 \leq 0,$$

contrary to hypothesis 3. Hence, the ρ_k are bounded, and the proof is complete. □

The proof of Theorem 17.6 is similar to that of Theorem 17.5 with the exception of inequality (17.18) resulting from hypothesis 2. In its place we reason as follows:

If $x \in M$, we have, by hypothesis 2A,

$$\begin{aligned} G(x) &\geq \|\dot{x}\|^2 - 2\int_{|x|<m} \alpha |x(t)|^2\,dt - C\int_{|x|>m} (|x|^q + 1)\,dt \\ &\geq \|\dot{x}\|^2 - 2\alpha\|x\|^2 - C(1 + m^{2-q} + m^{-q})\int_{|x|>m} |x|^q\,dt \\ &\geq \|\dot{x}\|^2(1 - [2\alpha T^2/4\pi^2]) - C'\int_{|x|>m} |x|^q\,dt \\ &\geq (1 - [\alpha T^2/2\pi^2])\|x\|_X^2 - C''\int_I \|x\|_X^q\,dt \\ &\geq (1 - [\alpha T^2/2\pi^2])\|x\|_X^2 - C'''\|x\|_X^q \\ &= \left(1 - [\alpha T^2/2\pi^2] - C'''\|x\|_X^{q-2}\right)\|x\|_X^2 \end{aligned}$$

by Lemma 17.8. Hence, we have

Lemma 17.10.

(17.40) $$G(x) \geq \varepsilon\|x\|_X^2, \quad \|x\|_X \leq \rho,\ x \in M,$$

for $\rho > 0$ sufficiently small, where $\varepsilon < 1 - [\alpha T^2/2\pi^2]$ is positive.

The remainder of the proof is essentially the same, but in this case $c > \varepsilon > 0$, obviating the need to consider the situation when $c = 0$.

In proving Theorem 17.7, we follow the proof of Theorem 17.5. In particular, it follows that $r > 0$. Moreover, (17.21) and (17.23) imply that

(17.41) $$\int_I H(t, x^{(k)}(t))\,dt \to -c.$$

On the other hand, by hypothesis 1A, we have

$$\begin{aligned} 0 &\leftarrow \|[\tilde{x}^{(k)}]^{\cdot}\|^2 - 2\int_I V(t, x^{(k)}(t))\,dt/\rho_k^2 \\ &\geq \|[\tilde{x}^{(k)}]^{\cdot}\|^2 - 2C\int_I (|\tilde{x}^{(k)}(t)|^2 + \rho_k^{-2})\,dt \\ &\to r^2 - 2C\int_I |\tilde{x}(t)|^2\,dt. \end{aligned}$$

Hence, $\tilde{x}(t) \not\equiv 0$. Let $\Omega_0 \subset I$ be the set on which $\tilde{x}(t) \neq 0$. The measure of Ω_0 is positive. Moreover, $|x^{(k)}(t)| \to \infty$ as $k \to \infty$ for $t \in \Omega_0$. Thus,

$$\int_I H(t, x^{(k)}(t))\,dt \leq \int_{\Omega_0} H(t, x^{(k)}(t))\,dt + \int_{I\setminus\Omega_0} W(t)\,dt \to -\infty$$

by hypothesis 4A. But this contradicts (17.41). Hence, the ρ_k are bounded, and the proof is complete.

17.4 Notes and remarks

The periodic, nonautonomous problem

$$\ddot{x}(t) = \nabla_x V(t, x(t)) \tag{17.42}$$

has an extensive history in the case of singular systems (cf., e.g., Ambrosetti–Coti Zelati [2]). The first to consider it for potentials satisfying (17.3) were Berger and the author [21] in 1977. We proved the existence of solutions to (17.42) under the condition that

$$V(t, x) \to \infty \quad \text{as} \quad |x| \to \infty$$

uniformly for a.e. $t \in I$. Subsequently, Willem [159], Mawhin [93], Mawhin–Willem [95], Tang [151], [152], Tang–Wu [154], [153], Wu–Tang [160] and others proved existence under various conditions (cf. the references given in these publications).

The periodic problem (17.1) was studied by Mawhin–Willem [96],[95], Long [88], Tang–Wu [155] and others (cf. the refernces quoted in them). Tang–Wu [155] proved the existence of solutions of problem (17.1) under the following hypotheses:

$$\text{(I)} \quad V(t, x) \to \infty \quad \text{as} \quad |x| \to \infty$$

uniformly for a.e. $t \in I$,

$$\text{(II)} \quad \exists\, a \in C(\mathbb{R}^+, \mathbb{R}^+), \quad b \in L^1(0, T, \mathbb{R}^+)$$

such that

$$|V(t, x)| + |\nabla V(t, x)| \leq a(|x|)b(t) \; \forall x \in \mathbb{R}^n \text{ and a.e. } t \in [0, T],$$

and

$$\text{(III)} \quad \exists\, 0 < \mu < 2, \quad M > 0$$

such that

$$\nabla V(t, x) \cdot x \leq \mu V(t, x) \; \forall\, |x| \geq M \text{ and a.e. } t \in [0, T].$$

Rabinowitz [103] proved existence under stronger hypotheses. In particular, he assumed

$$\text{(I}'\text{)} \quad \exists \text{ constants } a_1, a_2 > 0, \; \mu_0 > 1,$$

such that

$$V(t, x) \geq a_1|x|^{\mu_0} + a_2 \; \forall\, x \in \mathbb{R}^n \text{ and a.e. } t \in [0, T]$$

in place of (I), and

$$\text{(III}'\text{)} \quad \exists\, 0 < \mu < 2, \; M > 0$$

such that

$$0 < \nabla V(t, x) \cdot x \leq \mu V(t, x) \; \forall\, |x| \geq M \text{ and a.e. } t \in [0, T]$$

in place of (III). Mawhin–Willem [96] proved existence for the case of convex potentials, while Long [88] studied the problem for even potentials. They assumed that $V(t, x)$ is subquadratic in the sense that

$$\exists\, a_3 < (2\pi/T)^2 \text{ and } a_4$$

such that

$$|V(t,x)| \leq a_3|x|^2 + a_4 \ \forall x \in \mathbb{R}^n \text{ and a.e. } t \in [0,T].$$

Mawhin–Willem [95] also studied the problem for a bounded nonlinearity. Tang–Wu [155] also proved the existence of solutions if one replaces (I) with

$$\int_0^T V(t,x)\,dt \to \infty \text{ as } |x| \to \infty$$

and $V(t,x)$ is $\gamma-$ subadditive with $\gamma > 0$ for a.e. $t \in [0,T]$. All of these authors studied only the existence of solutions. Here, we studied the problem under much weaker assumptions and showed the existence of nonconstant solutions.

Little was done concerning nonconstant solutions of problem (17.1). For the homogeneous case, Ben Naoum–Troestler–Willem [18] proved the existence of a nonconstant solution. For the case $T = 2\pi$, Theorem 17.5, with substantially stronger hypotheses, was proved by Nirenberg (cf. Ekeland and Ghoussoub [60]). In place of hypothesis 2, they assumed

$$V(t,x) \leq \frac{3}{2\pi^2}, \quad |x| \leq 1,\ t \in \mathbb{R},\ x \in \mathbb{R}^n.$$

In place of hypotheses 3 and 4, they assumed the superquadraticity condition

$$V(t,x) > 0, \quad H_\mu(t,x) \leq 0, \quad |x| \geq C,\ t \in \mathbb{R},\ x \in \mathbb{R}^n,$$

for some $\mu > 2$, which implies these hypotheses and

$$V(t,x) \geq C|x|^\mu - C', \quad x \in \mathbb{R}^n, C > 0,$$

among other things.

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